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A Numerical Solution of a Convection-Dominated Equation Arising in Biology

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Abstract: The Reproducing Kernel Space Method (RKSM) is used to solve a convection-dominated equation, a singularly perturbed boundary value problem associated with biology science. The present method compared with the B-spline method (Lin *et al*., 2009), reveals that the present method is more effective and convenient.

Keywords: Convection-dominated equation, gram-schmidt orthogonal process, reproducing kernel

INTRODUCTION

The theory of singularly perturbed problems has become very important area of interest in the recent years. In these problems, a small parameter multiplies the highest order derivative and there exist boundary layers where the solutions change rapidly. It is a well known fact that the solution of singularly perturbed boundary-value problem exhibits a multiscale character (Doolan *et al*., 1980; Kevorkin and Cole, 1996; Miller *et al*., 1996). A reproducing kernel Hilbert space is a useful framework for constructing approximate solutions of differential equations (Akram and Rehman, 2011a, b).

The following convection-dominated equation associated with biology (Lin *et al*., 2009) can be considered as:

$$\left.\begin{aligned} -\varepsilon u^{(2)}(x) + u^{(1)}(x) + u(x) &= 1, \\ u(0) = 0,\ u(1) = 0, \quad 0 < x &\le 1 \end{aligned}\right\} \tag{1}$$

where,

$0< \varepsilon<1$, ε : A small positive parameter
f (x) : Continuous functions on [0, 1]

This problem arises in transport phenomena in chemistry and biology.

Let L be the differential operator and homogenization of the boundary conditions of system (1) can be transformed into the following form:

$$\left.\begin{aligned} Lu(x) &= f(x), \\ u(0) = u(1) = 0, \quad 0 < x &\le 1 \end{aligned}\right\} \tag{2}$$

The solution of system (2) provides the solution of the system (1).

METHODOLOGY

Reproducing kernel spaces: The reproducing kernel space $W^3{}_2[0,1]$ is defined by $W_2^3[01] = \{u(x) / u^{(i)}(x), i = 0, 1, 2$ are absolutely continuous real valued functions in [0,1], $u^{(3)}(x) \in L^2[0,1]\}$. The inner product and norm in $W^3{}_2[0,1]$ are given by:

$$< u(x), v(x) >= \int_0^1 (u^{(2)}(x)v^{(2)}(x) + u^{(3)}(x)v^{(3)}(x))\, dx \tag{3}$$

$$\|u(x)\| = \sqrt{< u(x), u(x) >}, \quad u(x), v(x) \in W_2^3[0,1] \tag{4}$$

Theorem 1: The space $W^3{}_2[0, 1]$ is a reproducing kernel Hilbert space. That is ,$\forall u(y) \epsilon W_2^3[0,1]$ and each fixed x, $y \epsilon [0, 1]$, there exists $R_x(y) \epsilon W^3{}_2[0, 1]$ such that $<u(y), R_x(y)> = u(x)$ and $R_x(y)$ is called the reproducing kernel function of space $W_2^3[0,1]$

The reproducing kernel function $R_x(y)$ is given by:

$$R_x(y) = \begin{cases} h(x,y) = \sum_{i=0}^{3} c_i y^i + c_4 e^y + c_5 e^{-y} & y \le x \\ h(y,x) = \sum_{i=0}^{3} d_i y^i + d_4 e^y + d_5 e^{-y}, & y > x \end{cases} \tag{5}$$

where,

$$h(x,y) = \frac{1}{6}\Big(-\frac{6e^{-y}(e^{2-x} + e^x + 2e^2(-1+x) - 2e\,x)}{2-2e^2}$$

$$-\frac{6(-1+e^{2-x} + e^x + e^2(-1+x) + x - 2e\,x)}{-1+e^2} + (-1+x)y^3 +$$

$$\frac{6e^y(-2+e^{2-x} + e^x + 2x - 2e\,x)}{2-2e^2} +$$

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$$\frac{e^{-x}(6e-6e^{2x}+e^{x}((-4+x)x(1+x)+e(-12+x(20+(-3+x)x))))y}{1+e})$$

The exact and approximate solutions: In the problem (2), the linear operator L: $W_2^3[0,1] \rightarrow W_2^1[0,1]$ is bounded. Using the adjoint operator L* of L and choose a countable dense subset T = {x_1, x_2, ..., x_n, ...}$\subset$ [0, 1] and let:

$$\varphi_i(y) = Q_{x_i}(y),\ i \in N \tag{6}$$

then

$$\psi_i(x) = L^*\varphi_i(x)$$

where,

$$\psi_i(x) \in W_2^3[0,1]$$

Lemma 1: $\{\boldsymbol{\Psi_i(x)}\}_{i=1}^{\infty}$ is a complete system of $W_2^3[0,1]$ and $\psi(x) = L_y R_x(y)|_{y=x_i}$ To orthonormalize the sequence $\{\Psi_i(x)\}_{i=1}^{\infty}$in the reproducing kernel space $W_2^3[0,1]$ Gram-Schmidt process can be used, as:

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \qquad i = 1,2,3,\ldots \tag{7}$$

Theorem 2: For all u(x) ϵ $W_2^3[0,1]$ the series

$$\sum_{i=1}^{\infty} < u(x), \overline{\psi}_i(x) > \overline{\psi}_i(x)$$

is convergent in the norm of $\|.\|_{W^3_2}$ On the other hand, if u(x) is the exact solution of the system (2) then:

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x) \tag{8}$$

Proof: Since u(x) ϵ $W_2^3[0,1]$ and can be expanded in the form of Fourier series about normal orthogonal system as:

$$u(x) = \sum_{i=1}^{\infty} < u(x), \overline{\psi}_i(x) > \overline{\psi}_i(x) \tag{9}$$

Since the space $W_2^3[0,1]$ is Hilbert space so the series

$$\sum_{i=1}^{\infty} < u(x), \overline{\psi}_i(x) > \overline{\psi}_i(x)$$

is convergent in the norm of $\|.\|_{W^3_2}$ From Eq. (7) and (9), it can be written as:

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} < u(x), \overline{\psi}_i(x) > \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} < u(x), \sum_{k=1}^{i}\beta_{ik}\psi_k(x) > \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} < Lu(x), \varphi_k(x) > \overline{\psi}_i(x) \end{aligned}$$

If u (x) is the exact solution of Eq. (2) and Lu = f(x), then:

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x)$$

The approximate solution obtained by the n-term intercept of the exact solution u (x), given by:

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x) \tag{10}$$

Numerical example:

Example (Lin *et al.*, 2009): The exact solution of the problem (1) is:

$$\begin{aligned} u(x) &= (e^{\lambda_2}-1)e^{\lambda_1 x}/(e^{\lambda_1}-e^{\lambda_2}) \\ &+ (1-e^{\lambda_1})e^{\lambda_2 x}/(e^{\lambda_1}-e^{\lambda_2}) + 1 \end{aligned}$$

where,

Table 1: Comparison of absolute error for the example between our method and the B-spline method (Lin *et al*., 2009), when n = 128

x	RKSM ε = 0.1	Lin *et al*. (2009) ε = 0.1	RKSM ε = 0.2	Lin *et al*. (2009) ε = 0.2	RKSM ε = 0.5	Lin *et al*. (2009) ε = 0.5	RKSM ε = 0.8	Lin *et al*. (2009) ε = 0.8
1/16	$1.05\text{x}10^{-4}$	0.0068	$2.31\text{x}10^{-5}$	0.0062	$2.55\text{x}10^{-6}$	0.0044	$6.181\text{x}10^{-7}$	0.0033
2/16	$1.59\text{x}10^{-4}$	0.0064	$3.97\text{x}10^{-5}$	0.0059	$4.71\text{x}10^{-6}$	0.0040	$1.145\text{x}10^{-6}$	0.0029
4/16	$1.83\text{x}10^{-4}$	0.0057	$5.13\text{x}10^{-5}$	0.0051	$7.99\text{x}10^{-6}$	0.0031	$1.603\text{x}10^{-6}$	0.0022
6/16	$1.91\text{x}10^{-4}$	0.0050	$5.91\text{x}10^{-5}$	0.0042	$9.97\text{x}10^{-6}$	0.0022	$1.988\text{x}10^{-6}$	0.0014
12/16	$1.85\text{x}10^{-4}$	0.0004	$6.67\text{x}10^{-5}$	0.0025	$8.37\text{x}10^{-6}$	0.0025	$2.532\text{x}10^{-6}$	0.0019
14/16	$1.63\text{x}10^{-4}$	0.0094	$5.85\text{x}10^{-5}$	0.0092	$5.07\text{x}10^{-6}$	0.0051	$2.177\text{x}10^{-6}$	0.0034

$$\lambda_1 = (1+\sqrt{1+4\varepsilon})/2\varepsilon, \lambda_2 = (1-\sqrt{1+4\varepsilon})/2\varepsilon$$

The comparison of the errors in absolute values between the method developed in this study and B-spline method (Lin *et al*., 2009) is shown in Table 1.

CONCLUSION

In this study, the reproducing kernel space method (RKSM) is developed for the solution singularly perturbed boundary value problem. The results obtained from our method are compared with the results obtained from the B-spline method (Lin *et al*., 2009) and found that present method gives better results. The results revealed that the method is a powerful mathematical tool for the solution of singularly perturbed boundary value problem. Numerical example also shows the accuracy of the method.

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