On the Nearest Weighted Point Theorem for Ranking Fuzzy Numbers

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Abstract: In this study the author discusses the problem of defuzzification by minimizing the weighted distance between two fuzzy quantities. Also, this study obtains the nearest point with respect to a fuzzy number and shows that this point is unique relative to the weighted distance. By utilizing this point, a method is presented for effectively ranking fuzzy numbers and their images to overcome the deficiencies of the previous techniques. Finally, several numerical examples following the procedure indicate the ranking results to be valid.

Keywords: Defuzzification, fuzzy number, ranking, weighted distance, weighted point

INTRODUCTION

Representing fuzzy numbers by proper intervals is an interesting and important problem. An interval representation of a fuzzy number may have many useful applications. By using such a representation, it is possible to apply some results, in fuzzy number approaches, derived in the field of interval number analysis. For example, it may be applied to a comparison of fuzzy numbers by using the order relations defined on the set of interval numbers. Various authors in (Grzegorzewski, 2002; Saneifard, 2009a) have studied the crisp approximation of fuzzy sets. They proposed a rough theoretic definition of that crisp approximation, called the nearest interval approximation of a fuzzy set. Moreover, quite different approach to crisp approximation of fuzzy sets was applied in Chakrabarty et al. (1998). They proposed a rough theoretic definition of that crisp approximation, called the nearest ordinary set of a fuzzy set. And they suggested a construction of such a set. They discussed rather discrete fuzzy sets. Their approximation of the given fuzzy set is not unique. Thus this study will not discuss this method.

Having reviewed the previous interval approximations, this study proposes here a method to find the weighted interval approximation of a fuzzy number that it is fulfills two conditions. In the first, this interval is a continuous interval approximation operator. In the second, the parametric distance between this interval and the approximated number is minimal.

PRELIMINARIES

The basic definitions of fuzzy number are given in Chu et al. (1994), Dubois and Prade (1987), Heilpern (1992) and Kaufmann and Gupta, 1991) as follows:

Definition 1: A fuzzy number A in parametric form is a pair \((L_A, R_A)\) of functions \(L_A(\alpha)\) and \(R_A(\alpha)\) that \(0 \leq \alpha \leq 1\), which satisfy the following requirements:

- \(L_A(\alpha)\) is a bounded monotonic increasing left continuous function
- \(R_A(\alpha)\) is a bounded monotonic increasing left continuous function
- \(L_A(\alpha) R_A(\alpha)\), \(0 \leq \alpha \leq 1\)

Definition 2: The trapezoidal fuzzy number \(A = (x_0, y_0, \sigma, \beta)\), with two defuzzifier \(x_0, y_0\) and left fuzziness \(\sigma > 0\) and right fuzziness \(\beta > 0\) is a fuzzy set where the membership function is as:

\[
A(x) = \begin{cases} 
\frac{1}{\sigma}(x - x_0 + \sigma) & \text{when } x_0 - \sigma \leq x \leq x_0 \\
1 & \text{when } x_0 \leq x \leq y_0 \\
\frac{1}{\beta}(y_0 - x + \beta) & \text{when } y_0 \leq x \leq y_0 + \beta \\
0 & \text{otherwise}
\end{cases}
\]

If \(x_0 = y_0\), then \(A = (x_0, \sigma, \beta)\) is called trapezoidal fuzzy number. The parametric form of triangular fuzzy number is \(L_A(\alpha) = x_0 - \sigma + \alpha \sigma, R_A(\alpha) = y_0 + \beta - \beta \alpha\).

Definition 3: For arbitrary fuzzy number \(A \in F\) (\(F\) denotes the space of fuzzy numbers) and \(0 \leq \sigma \leq 1\), function \(f: F \times [\sigma, 1] \rightarrow F\) such that \(f(A, \sigma) = (L_{\sigma A}, R_{\sigma A})\) is called delta – vicinity of the fuzzy number \(A\). Then there is:

\[
L_{\sigma A}(\alpha) = \begin{cases} 
L_A(\alpha) & \text{when } \alpha \in [0, \sigma] \\
L_A(\sigma) & \text{when } \alpha \in [\sigma, 1]
\end{cases}
\]

and
\[ R_{\delta}(\alpha) = \begin{cases} R_\delta(\alpha) & \text{when } \alpha \in [\delta, 1] \\ R_\delta(\delta) & \text{when } \alpha \in [0, \delta] \end{cases} \] (1)

If \( A = (x_0, y_0, \sigma, \beta) \) is a trapezoidal fuzzy number, the parametric form of it is \( A_\delta = (L_{\delta A}, R_{\delta A}) \) that is as follows:

\[ L_{\delta A}(\alpha) = \begin{cases} x_0 - \delta + \sigma \alpha, & \text{when } \alpha \in (\delta, 1] \\ x_0 - \delta + \sigma \delta, & \text{when } \alpha \in [0, \delta] \end{cases} \]

and

\[ R_{\delta A}(\alpha) = \begin{cases} y_0 + \beta - \sigma \alpha, & \text{when } \alpha \in (\delta, 1] \\ y_0 + \beta - \sigma \delta, & \text{when } \alpha \in [0, \delta] \end{cases} \] (2)

**Definition 4:** The following values constitute the weighted averaged representative and weighted width, respectively, of the fuzzy number \( A \):

\[ I(A) = \int_0^1 (cL_\alpha(\alpha) + (1 - c)R_\alpha(\alpha)) p(\alpha) d\alpha \]

\[ D(A) = \int_0^1 (R_\alpha(\alpha) - L_\alpha(\alpha)) p(\alpha) d\alpha \]

Here \( 0 \leq c \leq 1 \).

**Definition 5:** Saneifard (2010): For two arbitrary fuzzy numbers \( A = (L_\alpha, R_\alpha) \) and \( B = (L_\beta, R_\beta) \) the quantity:

\[ d_p(A, B) = \left( \int_0^1 f(\alpha)(L_\alpha(\alpha) - L_\beta(\alpha))^2 d\alpha + \int_0^1 f(\alpha)(R_\alpha(\alpha) - R_\beta(\alpha))^2 d\alpha \right)^{\frac{1}{2}} \] (3)

Was \( f : [0, 1] \rightarrow [0, 1] \) is a bi - symmetrical (regular) weighted, is called the bi - symmetrical (regular) weighted distance between \( A \) and \( B \) based on \( f \).

**Definition 6:** Grzegorzewski (2002): An operator \( I : F \rightarrow (\text{set of closed intervals in } R) \) is called an interval approximation operator if for any \( A \in F \):

\( (a') \) \( I(A) \subseteq \text{supp}(A) \)

\( (b') \) \( \text{Core}(A) \subseteq I(A) \)

\( (c') \) \( \forall(\epsilon > 0) \exists(\sigma > 0) \text{ s.t. } d(A, B) < \sigma \Rightarrow d(I(A), I(B)) < \epsilon \)

where, \( d : F \rightarrow [0, + \infty] \) denotes a metric defined in the family of all fuzzy numbers.

**Definition 7:** Grzegorzewski (2002): An interval approximation operator satisfying in condition \( (c') \) for any \( A, B \in F \) is called the continuous interval approximation operator.

NEAREST WEIGHTED INTERVAL OF A FUZZY NUMBER

Various authors in Grzegorzewski (2002) and Saneifard (2009b) have studied the crisp approximation of fuzzy sets. They proposed a rough theoretic definition of that crisp approximation, called the nearest ordinary set and nearest interval approximation of a fuzzy set. In this section, the researchers will propose another approximation called the weighted interval – value approximation. Let \( A = (L_A, R_A) \) be an arbitrary fuzzy number. This study will try to find a closed interval \( C_{d_p}(A) = [L_C, R_C] \), which is the weighted interval to \( A \) with respect to metric \( d_p \). So, this study has to minimize:

\[ D_p(A, C_{d_p}(A)) = \left( \int_0^1 f(\alpha)(L_\alpha(\alpha) - L_C)^2 + (R_\alpha(\alpha) - R_C)^2 d\alpha \right)^{\frac{1}{2}} \] (4)

with respect to \( L_C \) and \( R_C \). In order to minimize \( d_p \) it suffices to minimize:

\[ \nabla D_p(L_C, R_C) = \left( \frac{\partial D_p}{\partial L_C}, \frac{\partial D_p}{\partial R_C} \right) = 0 \]

Therefore; this study has the following equations:

\[ \begin{cases} \frac{\partial D_p}{\partial L_C} = -2 \int_0^1 f(\alpha)(L_\alpha(\alpha) - L_C) d\alpha = 0 \\ \frac{\partial D_p}{\partial R_C} = -2 \int_0^1 f(\alpha)(R_\alpha(\alpha) - R_C) d\alpha = 0 \end{cases} \] (5)

The parameters \( L_C \) associated with the left bound and \( R_C \) associated with the right bound of the nearest weighted interval can be found by using Eq. (5) as follows:

\[ L_C = \int_0^1 f(\alpha)L_\alpha(\alpha) d\alpha \]

\[ R_C = \int_0^1 f(\alpha)R_\alpha(\alpha) d\alpha \] (6)

**Remark 1:** Since det

\[ \begin{vmatrix} \frac{\partial D_p}{\partial L_C} & \frac{\partial D_p}{\partial L_C} \\ \frac{\partial D_p}{\partial R_C} & \frac{\partial D_p}{\partial R_C} \end{vmatrix} = \text{det} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 > 0 \]

therefore \( L_C \) and \( R_C \) given by (6), minimize \( (d_p(A, C_{d_p}(A)) \) Therefore, the interval:

\[ C_{d_p}(a) = \left[ \int_0^1 f(\alpha)L_\alpha(\alpha) d\alpha, \int_0^1 f(\alpha)R_\alpha(\alpha) d\alpha \right] \] (7)
is the nearest weighted interval approximation of fuzzy number \( A \) with respect \( d_p \). Now, suppose that this study wants to approximate a fuzzy number by a crisp interval. Thus the researchers have to use an operator \( C_{dp} \) which transforms fuzzy numbers into family of closed intervals on the real line.

**Lemma 1:** Saneifard (2009a):

**Theorem 1:** Saneifard (2009a): The operator \( C_{dp} \) is an interval approximation operator, i.e., \( C_{dp} \) is a continuous interval approximation operator.

\[
(2\int_0^1 f(\alpha)g(\alpha)d\alpha)^2 \leq 2\int_0^1 f(\alpha)g^2(\alpha)d\alpha
\]

**NEAREST WEIGHTED POINT OF A FUZZY NUMBER**

Let \( A = (L_A, R_A) \) be an arbitrary fuzzy number and \( C_{dp} (A) = [L_C, R_C] \), be the nearest weighted interval of it. If \( L_C = R_C \), then \( C_{dp} (A) = [L_C, R_C] = N_P (A) \), is the nearest weighted point approximation to fuzzy number \( A \) and its unique. The value of \( N_P (A) \) is as follows:

\[
N_P (A) = \int_0^1 f(\alpha)(L_A(\alpha) + R_A(\alpha))d\alpha
\]

The above equation introduces in the following Theorem.

**Theorem 1:** Let \( A = (L_A, R_A) \) be a fuzzy number and \( f(\alpha) \) be a bi symmetrical weighted function. Then \( N_P (A) \) is nearest weighted point to fuzzy number \( A \).

**Proof:** For the proof of Theorem it suffices that we replace \( L_C = R_C = N_P (A) \) in (4) and then minimize function \( D_p(A, N_P (A)) = d_p(A, N_P (A)) \) with respect to \( N_P (A) \). It is clear that, the parameter \( N_P (A) \) which minimizes Eq. (8) must satisfy in.

\[
\nabla D_p(A, N_P (A)) = (\frac{\partial D_p}{\partial N_P (A)}) = 0
\]

Therefore, this study has:

\[
\frac{\partial D_p(A, N_P (A))}{\partial N_P (A)} = -2\int_0^1 f(\alpha)
\]

\[
[L_A(\alpha) - N_P (A)]N_P + [R_A(\alpha) - N_P (A)]d\alpha = 0
\]

The solution is:

\[
N_P (A) = \int_0^1 f(\alpha)(cL_A(\alpha) + (1 + c)R_A(\alpha))d\alpha \quad (10)
\]

Since \( \frac{\partial D_P}{\partial N_P} = 2 > 0 \), therefore \( N_P (A) \) actually minimize \( D_p (A, N_P (A)) \) and simultaneously minimize \( d_p(A, N_P (A)) \).

**Theorem 2:** The nearest weighted point approximation to a given fuzzy number \( A \) is unique.

**Proof:** To prove the uniqueness of the operator \( N_P (A) \), we show that for any \( C \in R \), \( N_P (A) \) holds. We can write

\[
N_P (A) = \int_0^1 f(\alpha)[\lambda L_A(\alpha) + \lambda R_A(\alpha)]d\alpha + \mu N_P (A)
\]

Then

\[
N_P (A) = \int_0^1 f(\alpha)[\lambda L_A(\alpha) + \lambda R_A(\alpha) + \mu R_2(\alpha)]d\alpha 
\]

\[
\mu N_P (A) = \int_0^1 f(\alpha)[\lambda L_A(\alpha) + \mu R_2(\alpha)]d\alpha
\]

The last sentence of the above diction is zero, hence \( D_p (A, C) = D_p (A, N_P (A)) + 2(N_P (A) - C2 \) Consequently \( D_P A, C = D_P A, N_P (A) + 2(N_P (A) - C2 \) then we have \( D_p(A, C) \geq D_p (A, N_P (A)) \) which completes the proof of theorem.

**Remark 1:** Let \( A \) and \( B \) be two fuzzy numbers and \( \lambda \) and \( \mu \) be positive numbers. Then we have \( N_P (A) = \mu B = \lambda N_P (A) \pm \mu NB(B) \)

**Proof:** Let us suppose for all \( 0 \leq \alpha \leq 1 \), \( A = (L_A(\alpha), R_A(\alpha)) \) and \( B = (L_B(\alpha), R_B(\alpha)) \) and \( f(\alpha) \) is a bi symmetrical weighted function. Then,

\[
\lambda A \pm \mu B = \lambda L_A(\alpha) \pm \mu L_B(\alpha), R_A(\alpha) \pm \mu R_B(\alpha)
\]

Then

\[
N_P (A) = \int_0^1 f(\alpha)[\lambda L_A(\alpha) + \mu R_2(\alpha)]d\alpha 
\]

\[
N_P (A) = \int_0^1 f(\alpha)[\lambda L_A(\alpha) + \mu R_2(\alpha)]d\alpha + \mu N_P (A)
\]

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ORDERING OF FUZZY NUMBERS BY THE NEAREST WEIGHTED POINT

In this section, the researchers will propose the ranking of fuzzy numbers associated with the nearest weighted point approximation. Ever, the nearest weighted point can be used as a crisp approximation of a fuzzy number, therefore the resulting approximation is used to rank the fuzzy numbers. Thus, $N_p(.)$ is used to rank fuzzy numbers.

**Definition 1:** Let $A$, $B \in F$ are two fuzzy numbers and $N_p(A)$ and $N_p(B)$ are the nearest weighted point of them. Define the ranking of $A$ and $B$ by $N_p$ on $F$, i.e.

- $N_p(A) < N_p(B)$, if and only if $A < B$
- $N_p(A) > N_p(B)$, if and only if $A < B$
- $N_p(A) = N_p(B)$, if $A = B$

Via Theorem 1, the nearest weighted point approximation of the $\delta$-vicinity $A$ is as follows:

$$N_p(A_\delta) = \int_0^\infty f(\alpha)(L_\alpha(\alpha) + R_\alpha(\alpha))d\alpha$$

Then, this study formulates the order $>$ and $<$ as $A < B$ if and only if $A \sim B$, $A < B$ if and only if $A < B$ or $A \sim B$. This study considers the following reasonable axioms that Wang and Kerre (2001) proposed for fuzzy quantities ranking.

Let $I$ be an ordering method, $S$ the set of fuzzy quantities for which the method $I$ can be applied and $F$ a finite subset of $S$. If $I$ is applied to the fuzzy quantities then $A, B$ in $F$ satisfy that $A$ has a higher ranking than $B$, that is $A > B$ by $I$ on $F$.

A-B by $I$ on $F$ and $A \geq B$ by $I$ on $F$ are similarly interpreted. The following proposition shows the reasonable properties of the ordering approach $I$. Let be the set of fuzzy quantities for which the nearest point method can be applied and $F$ and $F'$ are two arbitrary finite subsets of $S$. Then we have:

**A-1:** For an arbitrary finite subset $A$ of $S$ and $A \in A$; $A \geq A$.

**A-2:** For an arbitrary finite subset $A$ of $S$ and $(A, B) \in A$; $A > B$ and $B > A$ results in $A \sim B$.

**A-3:** For an arbitrary finite subset $A$ of $S$ and $(A, B, C) \in A$; $A > B$ and $B > C$ produces $A > C$.

**A-4:** For an arbitrary finite subset $A$ of $S$ and $(A, B) \in A$; $\inf \ sup \ p(A) > \sup \ sup \ p(B)$; give $A > B$.

A'-4: For an arbitrary finite subset $A$ of $S$ and concludes as $A > B$.

**A-5:** Let $S$ and $S'$ be two arbitrary finite sets of fuzzy quantities in which $C_{dp}(.)$ can be applied, where both $A$ and $B$ belong to $S \cap S'$. Thus, the ranking order $A > B$ by $C_{dp}(.)$ on $S$ iff $A > B$ by $C_{dp}(.)$ on $S$ is achieved.

**A-6:** Let $A, B, A + C$ and $B + C$ be elements of $S$. If, $A > B$ then $A + C > B$ by $C_{dp}(.)$ on $\{A + C, B + C\}$.

A'-6: Let $A, B, A + C$ and $C + B$ be elements of $S$. If $A > B$ by $C_{dp}(.)$ on $A$ and $B$, then $A + C > B + C$ by $C_{dp}(.)$ on $\{A + C, B + C\}$.

**Remark 1:** Ranking order $N_p$ has the axioms $A_1, A_2, ..., A_6$.

**Proof:** The proof is similar to Saneifard (2010).

**Remark 2:** If $A < B$, then $-A > -B$.

Hence, this study can infer ranking order of the images of the fuzzy numbers.

**EXAMPLES**

In this section, we want to compare the proposed method with others. Saneifard (2009b):

**Example 1:** Consider the data used in Saneifard (2009b), i.e., the three fuzzy numbers, $A = (5, 6, 7)$, $B = (5.9, 6, 7)$ and $C = (6, 6, 7)$, as shown in Fig. 1.

According to Eq. (10) the ranking index values are obtained, i.e., $N_p(A) = 6$, $N_p(B) = 6.15$ and $N_p(C) = 6.16$. Thus, the ranking order of fuzzy numbers is $A > B > C$ However, by Chu and Tsao’s Approach (Chu and Tsao, 2002), the ranking order is $B > C > A$.
Table 1: Comparative results of example 1

<table>
<thead>
<tr>
<th>Fuzzy number</th>
<th>New approach</th>
<th>Sign distance with p = 1</th>
<th>Sign distance with p = 2</th>
<th>Chu-Tsao</th>
<th>Cheng distance</th>
<th>Cv Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>6</td>
<td>6.12</td>
<td>8.52</td>
<td>3.000</td>
<td>6.021</td>
<td>0.028</td>
</tr>
<tr>
<td>B</td>
<td>6.15</td>
<td>12.45</td>
<td>8.82</td>
<td>3.126</td>
<td>6.349</td>
<td>0.009</td>
</tr>
<tr>
<td>C</td>
<td>6.16</td>
<td>12.50</td>
<td>8.85</td>
<td>3.085</td>
<td>6.351</td>
<td>0.008</td>
</tr>
<tr>
<td>Results</td>
<td>A≺B≺C</td>
<td>A≺B≺C</td>
<td>A≺C≺B</td>
<td>A≺C≺B</td>
<td>A≺C≺B</td>
<td>A≺C≺B</td>
</tr>
</tbody>
</table>

Meanwhile, using the proposed CV index, the ranking order is A≻B≻C. From Fig. 1, it is obvious that the ranking results obtained by the existing approaches (Cheng, 1999; Chu and Tsao, 2002) are unreasonable and inconsistent. On the other hand, in Abbasbandy and Asady (2006), the ranking result is C≻B≻A which is the same as the one is obtained by the proposed method. However, the authors’ approach proves to be simpler in the computation procedure. Based on the analysis results from Abbasbandy and Asady (2006), the ranking results of this effort and other approaches are listed in Table 1.

**Example 2:** Consider the data used in Saneifard (2009b), i.e., the four fuzzy numbers, A = (-1, 0, 1), B = (2.2, 2.75, 3.5), C = (2, 3, 3.5) and D = (4, 5, 6). According to Eq. (10) the ranking index values are obtained, i.e., N_F (A) = 1, N_F (B) = 1.63, N_F (C) = 2.4 and N_F (D) = 5. The ranking order of fuzzy numbers is A≺B≺C≺D. But by the Z.-X. Wang method the ranking order is A≺C≺B≺D. Which is unreasonable and this is another shortcoming of the Z.-X. Wang method. In the real word, we can illustrate many examples like the above example again.

**CONCLUSION**

In this study, the authors proposed a defuzzification using minimize of the weighted distance between two fuzzy numbers and by using this defuzzification we proposed a method for ranking of fuzzy numbers. Roughly, there not much difference in our method and theirs. The method can effectively rank various fuzzy numbers and their images.

**REFERENCES**


