End-Regularity of the Join of n Split Graphs

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Abstract: A graph X is said to be End-regular if its endomorphism monoid End(X) is a regular semigroup. In this study, End-regular graphs which are the join of n split graphs are characterized. We give the conditions under which the endomorphism monoid of the join of n splits graphs is regular.

Keywords: Endomorphism, join of n graphs, regular semigroup, split graph

INTRODUCTION

Endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained. The aim of this research is to establish the relationship between graph theory and algebraic theory of semigroups and to apply the theory of semigroups to graph theory. Just as Petrich and Reilly pointed out in Petrich et al. (1999), in the great range of special classes of semigroups, regular semigroups take a central position from the point of view of richness of their structural “regularity”. So it is natural to ask for which graph G the endomorphism monoid of G is regular (such an open question raised in Marki (1988)). However, it seems difficult to obtain a general answer to this question. So the strategy for answering this question is to find various kinds of conditions of regularity for various kinds of graphs. In Wilkeit (1996) the connected bipartite graphs whose endomorphism monoids are regular were explicitly found. An infinite family of graphs with regular endomorphism monoids were provided in Li (2003) and the joins of two trees with regular endomorphism monoids were also characterized. Hou et al. (2008) explored the endomorphism monoid of the complement of a path p_n with n vertices. It was shown that the endomorphism monoid of the complement of a path is an orthodox semigroup. The split graphs and the join of split graphs with regular endomorphism monoids were studied in Li et al. (2001), Fan (1997) and Hou et al. (2012), respectively. The split graphs with orthodox endomorphism monoids were characterized in Fan (2002). In this study, we continue to explore the endomorphisms monoids of the joins of n split graphs and characterize such graphs whose endomorphism monoids are regular.

The graphs considered in this paper are finite undirected graphs without loops and multiple edges. Let X be a graph. The vertex set of X is denoted by V(X) and the edge set of X is denoted by E(X). The cardinality of the set V(X) is called the order of X. If two vertices x_1 and x_2 are adjacent in the graph X, the edge connecting x_1 and x_2 is denoted by {x_1, x_2} and write {x_1, x_2} ∈ E(X). For a vertex v of X, denote by N_X(v) the set {x ∈ V(X) | {x, v} ∈ E(X)} and called it the neighborhood of v in X, the cardinality of N_X(v) is called the degree or valency of v in X and is denoted by d_X(v). A subgraph H is called an induced subgraph of X if for any a, b ∈ H, {a, b} ∈ H if and only if {a, b} ∈ E(X). We denote by K_n a complete graph with n vertices. A clique of a graph X is the maximal complete sub graph of X. The clique number of X, denoted by σ(X), is the maximal order among the cliques of X. Let X_1, X_2, ..., X_n be n graphs. The join of X_1, X_2, ..., X_n, denoted by X_1 + X_2 + ... + X_n, is a graph with V(X_1 + X_2 + ... + X_n) = V(X_1) ∪ V(X_2) ∪ ... ∪ V(X_n) and E(X_1 + X_2 + ... + X_n) = E(X_1) ∪ E(X_2) ∪ ... ∪ E(X_n) ∪ \{\{a,b\} | a ∈ V(X_i), b ∈ V(X_j), (where i ≠ j)\}.

Let G be a graph. A subset K ⊆ V(G) is said to be complete if \{a, b\} ∈ E(G) for any two vertices a, b ∈ K. A subset S ⊆ V(G) is said to be independent if \{a, b\} ∉ E(G) for any two vertices a, b ∈ S. A graph X is called split graph if its vertex set V(X) can be partitioned into two disjoint (non-empty) sets S and K, such that S is an independent set and K is a complete set. We can always assume that any split graph X has a unique partition V(X) = K ∪ S, where K a maximal is complete set and S is an independent set. Since K is a maximal complete set of X, it is easy to see that for any y ∈ S, 0 ≤ d_X(y) ≤ n − 1, where n = σ(X).

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Lemma 1 (Li, 2003): Let X be a graph. Then X is End-regular if and only if X + K_r is End-regular for any n \geq 1.

The following are some known results about split graphs which are essential for our consideration.

Lemma 5 (Li et al., 2001): Let X be a connected split graph with V(X) = K \cup S, where S is an independent set and K is a maximal complete set, |K| = n. Then X is End-regular if and only if there exists r \in \{1, 2, \ldots, n-1\} such that \text{d}(x) = r for any x \in S.

Lemma 6 (Li et al., 2001): A non-connected split graph X is End-regular if and only if X exactly consists of a complete graph and several isolated vertices.

END-REGULAR JOINS OF n SPLIT GRAPHS

The End-regular split graphs have been characterized in Lemma 5 and 6. In this section, we will characterize the End-regular graphs which are the join of n split graphs.

Let X_i be a split graph with V(X_i) = V(K_i) \cup S_i, where, S_i = \{x_{i1}, \ldots, x_{ip_i}\} is an independent set and V(K_i) = \{k_{i1}, k_{i2}, \ldots, k_{ip_i}\} is a maximal complete set. Then the vertex set V(X_1 + X_2 + \ldots + X_n) of X_1 + X_2 + \ldots + X_n can be partitioned into n + 1 parts K, S_1, S_2, \ldots, S_n i.e., V(X_1 + X_2 + \ldots + X_n) is a complete n-partite graph. By Lemma 3, we know if X + Y is End-regular, then both of X and Y are End-regular. Clearly, if X_1 + X_2 + \ldots + X_n is End-regular, then X_i is End-regular for any 1 \leq i \leq n. So we always assume that X_i are End-regular split graphs in the sequel unless otherwise stated. Moreover, let d_i be the valency of the vertices of S_i in X_i. Clearly, if X_i is connected, then 0 \leq d_i \leq 1; if X_i is non-connected, then d_i = 0.

Lemma 7: If X_1 + X_2 + \ldots + X_n is End-regular, then q_i - d_i = q_i - d_i for any 1 \leq i \leq n and 1 \leq i \leq n.

Proof: Suppose that q_i - d_i \neq q_i - d_i. Then we have q_i + q_2 + \ldots + q_n - d_i \neq q_1 + q_2 + \ldots + q_n - d_i. Let q_1 + q_2 + \ldots + q_n - d_i < q_1 + q_2 + \ldots + q_n - d_i. As q_i < n, for any x \in S_i, x is not adjacent to exactly q_i - d_i vertices of V(K_i) in X_i, so x is not adjacent to exactly q_i - d_i vertices of V(K_i) in X_1 + X_2 + \ldots + X_n, take such a vertex and write k_x.

Let x_1 be a vertex of S_1 and y_1 be a vertex of S_1, since we have |V(K) \cap N(x_1)| = d_i + q_i - d_i + q_1 =
Let $X_1, X_2, \ldots, X_n$ be $n$ split graphs with $d_1 \leq q_1 - 2$, $d_1 - q_1 = q_j - d_j$ for any $1 \leq s \leq n$ and $1 \leq j \leq n$. Then for any endomorphism $f$ of $X_1 + X_2 + \ldots + X_n$, if $f$ is an induced subgraph of $X_1 + X_2 + \ldots + X_n$ and if only if there are no two vertices $x_1, x_2 \in S_i$ such that $N_{X_i}(x_1) \cup N_{X_i}(x_2) = V(K_i)$ for any $1 \leq i \leq n$.

**Proof:** Necessity follows from the proof of Lemma 8. Conversely, assume there are no two vertices $x_1, x_2 \in S_i$ such that $N_{X_i}(x_1) \cup N_{X_i}(x_2) = V(K_i)$ for any $1 \leq i \leq n$. As the proof of Lemma 8, it is easy to show that for any two vertices $x_1, x_2 \in S_i$, there is no endomorphism $f$ such that $f(x_1) \in S_i$ and $f(x_2) \in S_j$ for any $1 \leq i, j \leq n$ and $i \neq j$.

Let $f \in End(X)$ and let $a, b \in I_f$ with $\{a, b\} \in E(X_1 + X_2 + \ldots + X_n)$. We need to prove that there exist $c \in f^{-1}(a), d \in f^{-1}(b)$ such that $\{c, d\} \in E(X_1 + X_2 + \ldots + X_n)$. If both of $a$ and $b$ are in $f(V(K))$, then there exist two vertices $c \in f^{-1}(a), d \in f^{-1}(b)$ such that $\{c, d\} \in E(X)$ since $f(V(K)) = V(K)$. If exactly one of $a$ and $b$ is in $f(V(K))$, without loss of generality, assume that $a \in f(V(K)), b \notin f(V(K))$, then there exists a vertex $c \in f(V(K))$ such that $f(c) = a$. Suppose $\{c, v\} \notin E(X_1 + X_2 + \ldots + X_n)$ for any vertex $v \in f^{-1}(b)$, let $u \in f^{-1}(b)$, then $u$ is adjacent to exactly $q_1 + q_2 + \ldots + q_{k-1} + d_i$ ($1 \leq i \leq n$) vertices in $V(K)$. Since $x_1, x_2, \ldots, x_{q_1} = q_2 + \ldots + q_{k-1} + d_i$. So $b$ is adjacent to $f(x_1), f(x_2), \ldots, f(x_{q_1} + q_2 + \ldots + q_{k-1} + d_i)$, clearly $f(x_1), f(x_2), \ldots, f(x_{q_1} + q_2 + \ldots + q_{k-1} + d_i)$ are distinct. We get that $b$ is adjacent to $q_1 + q_2 + \ldots + q_{k-1} + d_i + 1$ vertices in $V(K)$, a contradiction. If both $a$ and $b$ are not in $f(V(K))$ and $\{c, d\} \notin E(X_1 + X_2 + \ldots + X_n)$ for any $c \in f^{-1}(a), d \in f^{-1}(b)$, then $f^{(1)}(a)$ and $f^{(1)}(b)$ are contained in the same $S_i$ ($i = 1, 2, \ldots, n$). From the discussion in the last paragraph, we have that $a = f^{(1)}(a)$ and $b = f^{(1)}(b)$ are in the same $S_i$ ($i = 1, 2$) and so $\{a, b\} \notin E(X_1 + X_2 + \ldots + X_n)$, a contradiction, as required.
**Proof:** Necessity follows immediately from Lemmas 7 and 8.

Conversely, let \( f \in \text{End}(X_1 + X_2 + \cdots + X_n) \). To show that \( f \) is regular, we need to prove that there exist two idempotents \( g \) and \( h \) in \( \text{End}(X) \) such that \( \rho_g - \rho_f \) and \( I_n = I_f \).

Since \( d_1 \leq n - 2 \) and \( d_2 \leq m - 2 \), \( f(V(K)) = V(K) \) and for any \( x \in S_1 \cup S_2 \cup \cdots \cup S_n \), there exists a vertex \( k_x \in V(K) \) such that \( x \) is not adjacent to \( k_x \). Let \( h \) be the mapping from \( V(X) \) to itself defined by:

\[
h(x) = \begin{cases} 
  x, & x \in f(X_1 + X_2 + \cdots + X_n), \\
  k_x, & \text{otherwise}.
\end{cases}
\]

Then \( h \in \text{End}(X) \) and \( h(V(K)) = V(K) \). If \( x \in f(X_1 + X_2 + \cdots + X_n) \), then \( h^2(x) = h(x) = x \); if \( x \in V(X_1 + X_2 + \cdots + X_n) \backslash f(X_1 + X_2 + \cdots + X_n) \), then it is easy to check that \( h^2(x) = h(k_x) = k_x \) \( h(x) = k_x = h(x) \) since \( k_x \in V(K) \subseteq f(X) \). Hence \( f \in Idpt(X_1 + X_2 + \cdots + X_n) \).

Clearly, \( I_f \) and \( I_h \) have the same set of vertices. Note that an idempotent endomorphism is half-strong. It follows from Lemmas 1 and 9 that both \( I_f \) and \( I_h \) are induced subgraph of \( X_1 + X_2 + \cdots + X_n \). Therefore \( I_n = I_f \).

Since \( f(V(K)) = V(K) \), \( [x]_{\rho_f} \) contains at most one vertex of \( V(K) \) for any \( x \in V(X_1 + X_2 + \cdots + X_n) \). Without loss of generality, we can suppose that \( V(X_1 + X_2 + \cdots + X_n) \) \( / \rho_i \) \( \rho_i \) \( s_i \) \( \rho_i \) \( n \).

Let \( g \) be a mapping from \( V(X_1 + X_2 + \cdots + X_n) \) to itself defined by:

\[
g(x) = \begin{cases} 
  k_i, & x \in [k_i]_{\rho_i}, \\
  s_i, & x \in [s_i]_{\rho_i}.
\end{cases}
\]

Then \( g \in \text{End}(X_1 + X_2 + \cdots + X_n) \). If any \( x \in [k_i]_{\rho_i} \), then \( g^2(x) = g(k_i) = k = g(x) \); if \( x \in [s_i]_{\rho_i} \), then \( g^2(x) = g(s_i) = s_i = g(x) \). Hence \( g^2 = g \). Clearly, \( \rho_g = \rho_f \) as required.

**Lemma 11:** Let \( X_1 + X_2 + \cdots + X_n \) be End-regular split graphs, \( d_i = q_i - 1 \) for any \( 1 \leq i \leq n \). Then \( X_1 + X_2 + \cdots + X_n \) is End-regular if and only if \( N_{X_i}(x_1) = N_{X_i}(x_2) \) for any \( x_1, x_2 \in S_i \) (where \( 1 \leq i \leq n \)).

**Proof:** Necessity follows immediately from Lemma 8. Conversely, since \( N_{X}(x_1) = N_{X}(x_2) \) for any \( x_1, x_2 \in S_i \), there is a unique vertex \( k_i \) in \( K_i \) such that \( \{x_1, x_2\} \in \text{End}(X_1 + X_2 + \cdots + X_n) \) for any \( x_1, x_2 \in S_i \). Now the subgraph induced by \( S_1 \cup S_2 \cup \cdots \cup S_n \cup \{k_1, k_2, \ldots, k_n\} \) is a complete \( n \) partite graph, denote it by \( T \). Hence \( X_1 + X_2 + \cdots + X_n \) is isomorphic to \( K_{q_1 + q_2 + \cdots + q_n} + T \). Since any complete \( n \) partite graph is End-regular, by Lemma 4, \( X_1 + X_2 + \cdots + X_n \) is End-regular.

Now we are ready for our main result in this section.

**Theorem 12:** Let \( X_1, X_2, \ldots, X_n \) be \( n \) split graphs. Then \( X_1 + X_2 + \cdots + X_n \) is End-regular if and only if:

- \( X_i \) is End-regular for any \( 1 \leq i \leq n \).
- \( q_i - d_i = q_j - d_j \) for any \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \).
- There are no two vertices \( x_1, x_2 \in S_i \) such that \( N(x_1) \cup N(x_2) = E(K_i) \) for any \( 1 \leq i \leq n \).

**Proof:** It follows directly from Lemmas 3, 5, 6, 10 and 11.

**END-ORTHODOX JOINS OF \( n \) SPLIT GRAPHS**

In this section, we will give the conditions under which the endomorphism monoids of the joins of the split graphs is orthodox.

**Lemma 13:** Let \( G_1, G_2, \ldots, G_n \) be \( n \) graphs. If \( G_1 + G_2 + \cdots + G_n \) is End-orthodox, then \( G_i \) is End-orthodox for any \( 1 \leq i \leq n \).

**Proof:** Since \( G_1 + G_2 + \cdots + G_n \) is End-orthodox, \( G_1 + G_2 + \cdots + G_n \) is End-regular. By Lemma 3, \( G_i \) is End-regular for any \( 1 \leq i \leq n \). To show \( G_i \) is End-orthodox, we only need to prove that the composition of any two idempotent endomorphisms of \( G_i \) is also an idempotent.

Let \( f_i \) and \( f_2 \) be two idempotents in \( \text{End}(G_i) \). Define two mappings \( g_1 \) and \( g_2 \) from \( V(G_1 + G_2 + \cdots + G_n) \) to itself by:

\[
g_1(x) = \begin{cases} 
  f_1(x), & x \in V(G_1) \\
  x, & \text{otherwise}
\end{cases}
\]

\[
g_2(x) = \begin{cases} 
  f_2(x), & x \in V(G_1 + G_2 + \cdots + G_n) \\
  x, & \text{otherwise}
\end{cases}
\]

Then \( g_1 \) and \( g_2 \) are two idempotents of \( \text{End}(G_1 + G_2 + \cdots + G_n) \) and so \( g_1g_2 \) is also an idempotent of \( \text{End}(G_1 + G_2 + \cdots + G_n) \) since \( G_1 + G_2 + \cdots + G_n \) is End-orthodox.
orthodox. Clearly, \( f_1 f_2 = (g_1 g_2)|_{G_1} \) the restriction of \( g_1 g_2 \) to \( G_1 \). Hence \( f_1 f_2 \) is an idempotent of \( \operatorname{End}(G_1) \) as required.

**Lemma 14:** Let \( G \) be a graph. Then \( G \) is End-orthodox if and only if \( G + K_n \) is End-orthodox for any positive integer \( n \).

**Proof:** If \( G + K_n \) is End-orthodox, then by Lemma 4, \( G \) is End-orthodox. Conversely, for any positive integer \( n \), by Lemma 4, if \( X \) is End-regular, then \( X + K_n \) is End-regular. Let \( f \) be an arbitrary idempotent of \( \operatorname{End}(X + K_n) \). Note that \( \sigma(G + K_n) = \sigma(G) + n \), \( V(K_n) \subseteq I_f \) and \( f|_{K_n} = 1|_{K_n} \), the identity mapping on \( K_n \). Hence \( f(V(G)) \subseteq V(G) \) and \( f|_G = 1|_G \). Let \( f_1 \) and \( f_2 \) are two idempotents of \( \operatorname{End}(G + K_n) \), let \( g_1 = f_1|G \) and \( g_2 = f_2|G \). Then \( g_1, g_2 \in \operatorname{Idpt}(G) \) and so \( g_1 g_2 \in \operatorname{Idpt}(G) \). Now \( (f_1 f_2)|_{K_n} = 1|_{K_n} \) and \((f_1 f_2)|G = g_1 g_2\) imply that \( f_1 f_2 \) is an idempotent of \( \operatorname{End}(G + K_n) \). Consequently \( G + K_n \) is End-orthodox.

Let \( X_i \) (\( i = 1, 2, \ldots, n \)) be two split graphs. If \( X_1 + X_2 + \ldots + X_n \) is End-orthodox, then \( X \) is End-orthodox if and only if:\n
\[ \forall x, y \in V(K) \text{ such that } 21{, } ( ) \neq x \wedge y, N_X(x) \neq N_X(y) \text{ for any two vertices } x, y \in V(K). \]

**Lemma 15:** Let \( X_1, X_2, \ldots, X_n \) be \( n \) split graphs with \( d \leq q_i - 2 \) for any \( 1 \leq i \leq n \). If \( N_X(x_1) \neq N_X(x_2) \) for any two vertices \( x_1, x_2 \in S_1 \cup S_2 \cup \ldots \cup S_n \), then \( f \in \operatorname{End}(X_1 + X_2 + \ldots + X_n) \) is a retraction (idempotents) if and only if:

- \( f(x) = x \) For any \( x \in V(K) \)
- For any \( y \in S_1 \cup S_2 \cup \ldots \cup S_n \), either \( f(y) \in V(K) \setminus N(y) \), or \( f(y) = y \)

**Proof:** Note that under the hypothesis of lemma \( X_1 + X_2 + \ldots + X_n \) has a unique maximum clique \( K \).

**Lemma 16:** Let \( X_1 + X_2 + \ldots + X_n \) be \( n \) split graphs with \( d \leq q_i - 2 \) for any \( 1 \leq i \leq n \). Then \( X_1 + X_2 + \ldots + X_n \) is End-orthodox if and only if:

- \( X_1 + X_2 + \ldots + X_n \) is End-regular
- \( N_X(x_1) \neq N_X(x_2) \) for any two vertices \( x_1, x_2 \in S_1 \cup S_2 \cup \ldots \cup S_n \)

**Proof:** Necessity is obvious. Conversely, since \( X_1 + X_2 + \ldots + X_n \) is End-regular, we only need to prove that the composition of two idempotent endomorphisms is also an idempotent. Let \( f \) be an arbitrary idempotent of \( \operatorname{End}(X_1 + X_2 + \ldots + X_n) \). Then \( f|_{V(K)} = V(K) \) and either \( f(x) = x \) or \( f(x) = k_x \) for any \( x \in S_1 \cup S_2 \cup \ldots \cup S_n \), where \( k_x \) is a vertex in \( V(K) \) such that \( \{x, k_x\} \notin E(X_1 + X_2 + \ldots + X_n) \). Now the assertion follows immediately.

**Lemma 17:** Let \( X_1, X_2, \ldots, X_n \) be \( n \) split graphs with \( d \leq q_i - 1 \) for any \( 1 \leq i \leq n \). Then \( X_1 + X_2 + \ldots + X_n \) is End-orthodox if and only if \( |S_1| = |S_2| = \ldots = |S_n| \).

**Proof:** Necessity is obvious. Conversely, \( X_1 + X_2 + \ldots + X_n \) is a join of a complete graph and a complete \( n \) partite graph. Since any complete \( n \) partite graph is End-orthodox, it follows from Lemma 14 that \( X_1 + X_2 + \ldots + X_n \) is End-orthodox.

**Theorem 18:** Let \( X_1 + X_2 + \ldots + X_n \) be \( n \) split graphs. Then \( X_1 + X_2 + \ldots + X_n \) is End-orthodox if and only if:

- \( X_1 \) is End-regular for any \( 1 \leq i \leq n \)
- \( q_i - d_i = q_j - d_j \) for any \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \)
- There are no two vertices \( x, y \in S_i \) such that \( N_X(x) \supseteq N_X(y) \) for any \( 1 \leq i \leq n \)
- \( N_X(s_i) \neq N_X(s_j) \) for any two vertices \( s_i, s_j \in S_1 \cup S_2 \cup \ldots \cup S_n \)

**Proof:** If \( X_1 + X_2 + \ldots + X_n \) is orthodox, then \( X_1 + X_2 + \ldots + X_n \) is regular and so both of \( X_i \) is regular for any \( 1 \leq i \leq n \). Now it follows immediately from Lemma 3, 16 and 17.

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