Approximate Solution of the Point Reactor Kinetic Equations of Average One-Group of Delayed Neutrons for Step Reactivity Insertion

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Abstract: The understanding of the time-dependent behaviour of the neutron population in a nuclear reactor in response to either a planned or unplanned change in the reactor conditions is of great importance to the safe and reliable operation of the reactor. In this study two analytical methods have been presented to solve the point kinetic equations of average one-group of delayed neutrons. These methods which are both approximate solution of the point reactor kinetic equations are compared with a numerical solution using the Euler's first order method. To obtain accurate solution for the Euler method, a relatively small time step was chosen for the numerical solution. These methods are applied to different types of reactivity to check the validity of the analytical method by comparing the analytical results with the numerical results. From the results, it is observed that the analytical solution agrees well with the numerical solution.

Key words: Analytical, Laplace transform, neutron population, point reactor kinetic, reactivity, time-dependent

INTRODUCTION

The base of a reactor model is a set of ordinary differential equations known as the point reactor kinetics equations. These equations which expresses the time-dependence of the neutron population and the decay of the delayed neutron precursors within a reactor are first order and linear, and essentially describe the change in neutron population within the reactor due to a change in reactivity.

One of the important properties in a nuclear reactor is the reactivity, due to the fact that it is directly related to the control of the reactor. For safety analysis and transient behavior of the reactor, the neutron population and the delayed neutron precursor concentration are important parameters to be studied. An important property of the kinetics equations is the stiffness of the system. The stiffness is a severe problem in a numerical solution of the point kinetics equations and results in the need for small time steps in a computational scheme.

A numerical algorithm (CORE) has been developed by Quintero-Leyva (2008) to solve the point kinetics equations. Kinard and Allen (2004) described an efficient numerical solution of the point kinetic equation in nuclear reactor dynamics. Power series solution (PWS) method for solving the point kinetics equations with lumped model temperature and feedback have also been developed and applied by Sathiyasheela (2009). Aboanber and Hamada (2003) solved the point reactor kinetic equations analytically in the presence of Newtonian temperature feedback for different types of reactivity input using a straightforward recurrence relation of a power series. Fan Zhang et al. (2008) described an analytic method study of point reactor kinetic equation when cold start-up. Daniel et al. (2009) presented analytical solution of the point kinetic equations for linear reactivity variation during the start-up of a nuclear reactor.

It is worthwhile trying to develop schemes for the solution of the point reactor kinetic equations. The importance of these schemes is that any mathematical or physical intuition gained from such schemes at this rather elementary level can usually be applied to more realistic models. The present study is concerned with the development of an approximate method for solving the point reactor kinetic equations of average one-group of delayed neutrons using a simple analytical method and Laplace transformation method. These methods are applicable to a general class of problems for different types of reactivity input.

METHODOLOGY

The simplified analytical solution of point reactor kinetic equations: The point reactor kinetics equations with average one-group of delayed neutrons (Hetrick, 1971; Glasstone and Sesonske, 1981) are given as follows:

\[
\frac{dn(t)}{dt} = \frac{\beta(t) - \beta}{\lambda} m(t) + \lambda C(t)
\]  (1)

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\[
\frac{dC(t)}{dt} = \frac{\beta}{\lambda} n(t) - \lambda C(t)
\]

(2)

where \(n(t)\) is the time dependent neutron population, \(C(t)\) is the average density of delayed neutron precursors, \(\rho(t)\) is the time dependent reactivity function, \(\beta\) is the total fraction of delayed neutrons, \(\lambda\) is the decay constant of delayed neutron precursors, \(\Lambda\) is the prompt neutrons generation time and \(t\) is time.

Rewriting Eq. (1) and (2) in a matrix form as:

\[
\frac{d}{dt} \begin{bmatrix} n(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} \frac{\rho - \beta}{\lambda} & \lambda \\ \frac{\beta}{\lambda} & -\lambda \end{bmatrix} \begin{bmatrix} n(t) \\ C(t) \end{bmatrix}
\]

(3)

The solution of Eq. (3) subjected to the initial conditions:

\[
\begin{cases}
 n(0) = n_0 \\
 C(0) = \frac{\beta n_0}{\lambda \Lambda}
\end{cases}
\]

(4)

where \(n(0)\) is the neutron population at time, takes the form:

\[
\begin{bmatrix} n(t) \\ C(t) \end{bmatrix} = \exp \left( \begin{bmatrix} \frac{\rho - \beta}{\lambda} & \lambda \\ \frac{\beta}{\lambda} & -\lambda \end{bmatrix} t \right) \begin{bmatrix} n_0 \\ \frac{\beta n_0}{\lambda \Lambda} \end{bmatrix}
\]

(5)

The exponential function of the matrix:

\[
\exp \left( \begin{bmatrix} \frac{\rho - \beta}{\lambda} & \lambda \\ \frac{\beta}{\lambda} & -\lambda \end{bmatrix} t \right)
\]

can be rewritten in the form:

\[
\exp \left( \begin{bmatrix} \frac{\rho - \beta}{\lambda} & \lambda \\ \frac{\beta}{\lambda} & -\lambda \end{bmatrix} t \right) = a_1(t)I + a_2(t) \begin{bmatrix} \frac{\rho - \beta}{\lambda} & \lambda \\ \frac{\beta}{\lambda} & -\lambda \end{bmatrix}
\]

(6)

where \(I\) is a 2 x 2 unit matrix.

Equation (6) can further be expressed as:

\[
\exp \left( \begin{bmatrix} \frac{\rho - \beta}{\lambda} & \lambda \\ \frac{\beta}{\lambda} & -\lambda \end{bmatrix} t \right) = \begin{bmatrix} a_1(t) & 0 \\ 0 & a_2(t) \end{bmatrix}
\]

(7)

To obtain the coefficients \(a_1(t)\) and \(a_2(t)\), Eq. (14) and (15) are solved as follows:

Subtracting Eq. (15) from (14) we get:

\[
\exp(\omega_1 t) - \exp(\omega_2 t) = \alpha_3(t)[(\omega_1 t) - (\omega_2 t)]
\]

(16)

and

\[
\exp(\omega_1 t) - \exp(\omega_2 t) = \alpha_3(t)[(\omega_1 t) - (\omega_2 t)]
\]

(17)
\( \alpha_2(t) = \frac{\exp(\omega_2t) - \exp(\omega_1t)}{\omega_2 - \omega_1} \) \hspace{1cm} (17)

Substitute Eq. (17) into (15) to get:

\[ \exp(\omega_2t) = \alpha_1(t) + \frac{\exp(\omega_1t) - \exp(\omega_2t)}{\omega_1 - \omega_2} \] \hspace{1cm} (18)

Rearrangement of Eq. (18) leads to:

\[ \alpha_1(t) = \exp(\omega_1t) - \omega_2 \left[ \frac{\exp(\omega_1t) - \exp(\omega_2t)}{\omega_1 - \omega_2} \right] \] \hspace{1cm} (19)

Solving Eq. (19) for \( \alpha_1(t) \), we get:

\[ \alpha_1(t) = \frac{\omega_1 \exp(\omega_1t) - \omega_2 \exp(\omega_2t)}{\omega_1 - \omega_2} = \frac{\omega_1 \exp(\omega_1t) - \omega_2 \exp(\omega_2t)}{\omega_1 - \omega_2} \] \hspace{1cm} (20)

and finally,

\[ \alpha_1(t) = \frac{\omega_1 \exp(\omega_1t) - \omega_2 \exp(\omega_2t)}{\omega_1 - \omega_2} \] \hspace{1cm} (21)

Equation (17) and (21) are substituted into Eq. (11) and (13) to obtain the final analytical expression for the neutron population and the average density for delayed neutron precursors as:

\[ n(t) = \frac{\omega_1 \exp(\omega_1t) - \omega_2 \exp(\omega_2t)}{\omega_1 - \omega_2} \left[ \rho_0 \exp(\omega_2t) - \rho_0 \exp(\omega_1t) \right] \] \hspace{1cm} (22)

and

\[ C(t) = \frac{\beta \rho_0}{\lambda^2} \left[ \frac{\omega_1 \exp(\omega_1t) - \omega_2 \exp(\omega_2t)}{\omega_1 - \omega_2} \right] \] \hspace{1cm} (23)

Laplace transform solution of the point kinetic equations: In this section, approximate solution of the point neutron kinetic equations for one group of delayed neutron is provided using Laplace transform.

From the point neutron kinetic Eq. (1) and (2), at \( t = 0 \), and introducing \( \rho_0 \) which is constant over time, by Laplace transforming or assuming an exponential time dependence, \( e^{st} \) form of the solution, the equations for the time dependent parts of the neutron population and the delayed neutron precursors are:

\[ s \rho_0 = n(s) + \lambda C(s) \] \hspace{1cm} (24)

\[ sC(s) = \frac{\beta}{\lambda} n(s) - \lambda C(s) \] \hspace{1cm} (25)

which can be reduced to:

\[ n(s) = \frac{f(s,n_0,C_0)}{Y(s)} \] \hspace{1cm} (26)

where \( Y(s) = \rho_0 - s \left( \Lambda + \frac{\beta}{s + \Lambda} \right) \) \hspace{1cm} (27)

The roots of \( Y(s) = 0 \), determine the time dependence of the neutron and precursor populations. \( Y(s) = 0 \) is known as the inverse hour or the in hour equation.

The roots of the reduced in hour equation is:

\[ s^2 \left( \frac{\rho - \beta}{\Lambda} - \lambda \right) - \frac{\rho \lambda}{\Lambda} = 0 \] \hspace{1cm} (28)

which has the solution:

\[ s_{1,2} = \frac{1}{2} \left( \frac{\rho - \beta}{\Lambda} - \lambda \pm \sqrt{\frac{1}{4} \left( \frac{\rho - \beta}{\Lambda} - \lambda \right)^2 + \frac{\rho \lambda}{\Lambda}} \right) \] \hspace{1cm} (29)

The assumed \( e^{st} \) time dependence, when used in Eq. (1) and (2) requires that for each of the roots, \( s_1 \) and \( s_2 \), there is a fixed relation between the precursor and neutron population as:

\[ \frac{C(t)}{n(t)} = \frac{\beta}{\Lambda(s_{1,2} + \lambda)} = -\left( \frac{\rho - \beta}{\Lambda} - s_{1,2} \right) / \lambda \] \hspace{1cm} (30)

This means that the solution of Eq. (1) and (2) is of the form:

\[ n(t) = A_1 e^{s_1t} + A_2 e^{s_2t} \] \hspace{1cm} (31)

and

\[ C(t) = \frac{\beta}{\Lambda(s_1 + \lambda)} e^{s_1t} + \frac{\beta}{\Lambda(s_2 + \lambda)} e^{s_2t} \] \hspace{1cm} (32)

Assuming that \( |\rho - \beta|/\Lambda >> \lambda \), the roots of Eq. (28) are:

\[ s_1 = \frac{\rho - \beta}{\Lambda}, \quad s_2 = \frac{\rho \lambda}{\rho - \beta} \] \hspace{1cm} (33)

The constants \( A_1 \) and \( A_2 \) can be evaluated by requiring that the solution satisfy the initial condition at \( t = 0 \) that is determined by setting \( \rho = 0 \) in Eq. (31) and (32) given:

\[ A_1 = \frac{n_0 \rho}{\rho - \beta} \quad \text{and} \quad A_2 = -\frac{n_0 \beta}{\rho - \beta} \] \hspace{1cm} (34)

where \( n_0 \) is the initial neutron population before the reactivity insertion and at \( t = 0 \), before reactivity insertion,
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
n(t) n(t)
0 1 0 2 0 3 0 4 0 5 0 6 0 7 0 8 0 9 0 1 0
Time (sec)
SAM
LTM
Euler method

Fig. 1: The decay of neutron population following a negative reactivity insertion into a critical reactor

Fig. 2: The increase in neutron population following a positive reactivity insertion into a critical reactor

\[ C_0 = \frac{\beta n_0}{\lambda}. \]

Thus, the solutions to Eq. (31) and (32) become:

\[ n(t) = n_0 \left[ \frac{\rho}{\rho - \beta} \exp \left( \frac{\beta - \rho}{\lambda} t \right) - \frac{\beta}{\rho - \beta} \exp \left( - \frac{\lambda \rho}{\rho - \beta} t \right) \right] \]  (35)

and

\[ C(t) = n_0 \left[ \frac{\rho \beta}{(\rho - \beta)^2} \exp \left( \frac{\rho - \beta}{\lambda} t \right) + \frac{\beta}{\lambda \beta} \exp \left( - \frac{\lambda \rho}{\rho - \beta} t \right) \right] \]  (36)

RESULTS AND DISCUSSION

In order to verify the analytical approximation method for neutron population presented in this paper, the Euler’s method was used for numerical solution of the point reactor kinetic Eq. (1) and (2). To obtain accurate solution for the Euler method, a relatively small time step of 0.001 s was chosen for the numerical solution of the point reactor kinetic equations.

The nuclear parameters used in the verification of the solution represented by Eq. (22) and (35) against the numerical method (Euler’s method) are \( \lambda = 0.08 \text{s}^{-1}, \lambda = 6.0 \times 10^{-5} \text{s}, \) and \( \beta = 0.0075. \)

Figure 1 shows the neutron decay following a step negative reactivity insertion \( \rho < 0 \) (i.e., large negative reactivity insertion, \( \rho = -0.05 \)) such as might be produced by scramping (or rapid insertion) of a control rod bank.

From the Fig. 1, it is observed that the neutron population drops promptly from \( n_0 \) to \( n_0/(1 - \rho/\beta) \) (i.e., the prompt jump) and then slowly decays. The result indicates that scramping (or rapid insertion) of a control rod bank cannot immediately shutdown a nuclear reactor. In other words, scramping cannot immediately reduce the neutron population or fission rate to near zero.

Figure 2 shows a step positive reactivity insertion, \( 0 < \rho < \beta, \) where \( \rho = 0.0015 \) such as might occur as a result of withdrawal of control rods.

It is observed from Fig. 2 that the neutron population increases promptly from \( n_0 \) to \( n_0/(1 - \rho/\beta) \) (i.e., the prompt jump), as the prompt neutron population adjust to the supercritical condition of the reactor. The neutron population then slowly increases mainly due to the rate of increase of delayed neutrons. The slow rate of increase of the neutron population following the prompt jump allows operators time for corrective control action to be taken before the fission rate becomes excessive.

CONCLUSION

An analytical solution has been presented in this paper to seek predicting the neutron population during withdrawal and insertion of control rods. The formulation consists of solving the point reactor kinetic equations of average one-group of delayed neutrons and considering prompt jump approximation. Two analytical methods have been presented, a simplified analytical method and Laplace transformation method.

Calculations were performed using the analytical methods and the results compared with a numerical solution using Euler’s method. To obtain accurate solution for the Euler method, a relatively small time step of 0.001 s was chosen. The comparison has shown that the analytical solution agrees well with the numerical solution.

REFERENCES


