Coapproximation in Probabilistic Normed Spaces

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Abstract: In this article, we studied the concept of best coapproximation in probabilistic normed spaces. We defined the best coapproximation on these spaces and generalized some definitions such as set of P-best coapproximation, P-coproximinal set, P-coapproximately compact and orthogonality relative to any set and then proved some interesting theorems.

Key words: P-best coapproximation, P-coproximinal, P-coapproximately compact and orthogonality relative to any set

INTRODUCTION

Menger (1942) introduced the notion of probabilistic metric spaces. The idea of Menger was to use distribution function in stead of non negative real numbers as values of the metric. The concept of probabilistic normed spaces (briefly, PN-spaces) was introduced by Sertnev (1963). Ghazal (2010) considered the best approximation and coapproximation, coapproximation in probabilistic normed spaces. In the sequel after an introduction to probabilistic normed spaces, we define the concept of best coapproximation in probabilistic normed space and generalized some definitions such as set of best coapproximation, coproximinal set and coapproximatively compact set.

Chang et al. (2001) defined some notions in probabilistic normed spaces as follows:

A distance distribution function (briefly, d.d.f.), is a function $F$ defined from extended interval $[0, +\infty]$ into the unit interval $I = [0,1]$, that is non decreasing and left continuous on $(0, +\infty)$ such that $F(0) = 0$ and $F(+\infty) = 1$. The family of all d.d.f.s will be denoted by $D^+$. We shall denote the distribution function $F_t(x)$ by $F_t(x)$, for each $x \in V$.

$D^+ = \{F \in D^+ | \lim_{t \to 0^+} F(t) = 1\}$ and $D^0 = \{F \in D^+ | F'(1)\neq \varnothing\}$.

By setting $F \leq G$ whenever $F(t) \leq G(t)$, for all $t \in \mathbb{R}^+$, one introduces a natural ordering in $D^+$. If $a \in \mathbb{R}^+$ then $H$ will be an element of $D^+$, defined by $H(t) = 0$ if $t \leq 0$ and $H(t) = 1$ if $t > 0$. It is obvious that $H \leq F$ if $t > 0$ for all $F \in D^+$.

A t-norm $\tau$ is a two place function $\tau : I \times I \to I$ which is associative, commutative, non decreasing in each place and such that $\tau(a, 1) = a$, for all $a \in [0, 1]$.

A triangle function is a mapping $(T : D^+ \times D^+ \to D^+)$ which is associative, commutative, non decreasing and for which $H$ is the identity, that is, $T(H,F) = F$, for every $F \in D^+$.

Definition 1: A probabilistic normed space (shortly, PN-space) is an ordered pair $(V, \Delta)$, where $V$ is a real linear space, $\Delta$ is a mapping from $V$ into $D^+$. (We shall denote the distribution function $\Delta(x)$ by $F_x$, $\forall x \in V$) satisfying the following conditions:

(PN-1) $F_x(t) = 1$ for all $t > 0$ if and only if $x = 0$
(PN-2) $F_x(0) = 0$
(PN-3) $F_{x+}(t) = F_x(t|\alpha|)$, for every $t > 0$, $\alpha \neq 0$ and $\alpha \in \mathbb{R}$
(PN-4) $F_{x+y}(t_1) = 1$, $F_{x+y}(t_2) = 1$, then $F_{x+y}(t_1+t_2) = 1$ for all $x,y \in V$.

Definition 2: A Menger PN-space is a triple $(V, \Delta, \tau)$, where $V$ is a PN-space and $\tau$ is a t-norm satisfy the following conditions:

(PN-4m) $F_{x+y}(t_1+t_2) \geq \tau (F_{x+y}(t_1), F_{x+y}(t_2))$
for all $x,y \in V$ and $t_1, t_2 \in \mathbb{R}^+$

Theorem 3: Let $(V, \Delta)$ be a PN-space and $\delta$ take its values in $D^+$. If we define the functions $||.||$ and $||.||_\alpha$, $\alpha \in (0, 1]$, as follows, respectively:

$||x|| = \inf_{t \geq 0} t$ if $F_x(t) \leq 1$  
$||x||_\alpha = \inf_{t \geq 0} t$ if $F_x(t) > 1-\alpha$

for all $x \in V$ and $\alpha \in (0, 1]$, then:
Theorem 4: Let \((V, \mathcal{S})\) be a PN-space and \(\mathcal{S}\) satisfy the following condition:

\[(\text{PN-B}): \text{For any } x, y \in V \text{ and } t_1, t_2 > 0, \lambda > 0, \text{ if } F_x(t_1) > 1-\lambda \text{ and } F_y(t_2) > 1-\lambda, \text{ then } F_{x+y}(t_1 + t_2) > 1-\lambda. \text{ Then:}\]

- For each \(a \in (0, 1]\), the function \(\rho_a(x)\) defined by:
  
  \[
  \rho_a(x) = \inf \left\{ t \geq 0 : F_x(t) > 1 - a \right\}
  \]

is a semi-norm and the family of semi-norms \(\{\rho_a, a \in (0, 1]\}\) can separate the points of \(V\), i.e., \(x = 0\), then there exists an \(a \in (0, 1]\) such that \(\rho_a(x) > 0\),

- \((V, \mathcal{S})\) is a locally convex topological space induced by the family of semi-norms \(\{\rho_a, a \in (0, 1]\}\) and, for each \(x \in V\)

\[
\mathcal{U}(x) = \{U(x, \alpha_1, \alpha_2, ..., \alpha_n, \varepsilon): \varepsilon > 0, \alpha_1, \alpha_2, ..., \alpha_n \in (0, 1], n \text{ is any positive integer} \}
\]

is the basis of neighborhoods of \(x\), where:

\[
\mathcal{U}(x, \alpha_1, \alpha_2, ..., \alpha_n, \varepsilon) = \{y \in V: \rho_{\alpha_i}(x-y) < \varepsilon, \alpha_i \in (0, 1], i = 1, ..., n\}
\]

The topology induced by the basis \(\mathcal{U}(x)\) of neighborhoods of \(x\) coincides with the topology induced by the following basis of neighborhoods of \(x\):

\[
\mathcal{N}(x) = \{N(\varepsilon, \alpha): \varepsilon > 0, \alpha > 0\}, \alpha \in (0, 1]\}
\]

where

\[
N_{\varepsilon}(x, \alpha) = \{y \in V: F_x(y) > 1 - \alpha\}
\]

**P-BEST CO ApproxIMATION IN PROBABILISTIC NORMED SPACE**

**Definition 1:** Let \(A\) be a nonempty subset of a PN-space \((V, \mathcal{S})\). For \(t > 0\), an element \(a_0 \in A\) is called a P-best coapproximation to \(x \in V\) from \(A\) if for every \(a \in A\):

\[
F_{a_0-a}(t) \geq F_{x-a}(t)
\]

The set of all such elements \(a_0\) that called a \(P\)-best coapproximation to \(x \in V\), is denoted by \(R'_A(x)\), i.e.,

\[
R'_A(x) = \{a_0 \in A: F_{a_0-a}(t) \geq F_{x-a}(t) \text{ for all } a \in A, t > 0\}.
\]

Putting

\[
\tilde{A} = \{x \in V: F_x(t) \geq F_{a_0}(t) \text{ for all } a \in A, t > 0\} = (R'_A)^{-1}(\{0\}),
\]

it is clear \(a_0 \in R'_A(x)\) if and only if \(x-a_0 \in \tilde{A}\).

**Definition 2:** Let \((V, \mathcal{S})\) be a PN-space. For \(t > 0\), the nonempty subset \(A \subseteq V\) is called P-coapproximal set if \(R'_A(x)\) is non-void for every \(x \in V\) and \(A\) is called P-coChebyshev set if for every \(x \in V\) the set \(R'_A(x)\) contains exactly one element.

**Definition 3:** Let \(A\) be a nonempty subset of a PN-space \((V, \mathcal{S})\), and \(\{x_n\}\) be a sequence of \(V\).

- Then the sequence \(\{x_n\}\) is said to be \(P\)-convergent to \(x \in V\) and denoted by \(x_n \xrightarrow{P} x\), if \(\lim_{n \to \infty} F_{x_n-x}(t) = 1\) for all \(x \in V\) and \(t > 0\).

- The set \(A\) is closed if and only if, whenever \(\{a_n\}\) is a sequence of points of \(A\) that \(P\)-converging to \(x \in V\), then \(x\) is also in \(A\). Also we recall that a subset \(A \subseteq X\) is called compact if each sequence \(\{a_n\}\) in \(A\) has a subsequence \(\{a_{n_k}\}\) that \(P\)-converging to an element \(a_0 \in A\).

**Theorem 4:** Let \(A\) be a nonempty subset of a PN-space \((V, \mathcal{S})\). Then for \(t > 0\):

- \(R'_A(x+y) = R'_A(x)+y\), for every \(x, y \in V\).

- \(R'_A(\alpha x) = \alpha R'_A(x)\), for every \(x \in V\) and any scaler \(\alpha \in R\setminus\{0\}\).

A is P-coapproximal (respectively P-coChebyshev) if and only if \(A+y\) is P-coapproximal (respectively P-coChebyshev) for every \(y \in V\).

**Proof:**

- For any \(x, y \in V\) and \(t > 0\), let \(a_0 \in R'_A(x+y)\) if and only if, \(F_{a_0-a}(t) \geq F_{x+y-a}(t)\) for all \((a+y) \in A+y\) if and only if, \(F_{(a_0-a)+y}(t) \geq F_{a+y}(t)\) for all \(a \in A\) if and only if, \((a_0-y) \in R'_A(x)\) i.e., \(a_0 \in R'_A(x+y)\).

- Let \(a_0 \in R'_A(\alpha x)\), for any \(x \in V\), \(t > 0\) and \(\alpha \in R\setminus\{0\}\) if and only if, \(F_{a_0-a}(t) \geq F_{a_0-y}(t)\) for all \(a \in A\) if and only if, \(F_{(a_0-a)+y}(t) \geq F_{a+y}(t)\) if and only if, \(\frac{1}{\alpha} a_0 \in R'_A(x)\) if and only, \(a_0 \in \alpha R'_A(x)\). Therefore, \(R'_A(\alpha x) = \alpha R'_A(x)\).

Is an immediate consequence of (i).
Theorem 5: Let \((V, \mathcal{V}, \tau)\) be a Menger PN-space and \(A\) be a convex subset of \(V\). Then for \(t > 0\) and \(x \in V, R_t'(x)\) is a convex subset of \(A\) for \(R_t'(x) \neq \emptyset\).

Proof: Let \(a_t, a_0 \in R_t'(x), t > 0\), and \(x \in V\), then:
\[
F_{a_t-a_0}(t) \geq F_{a_0}(t) \quad \text{and} \quad F_{a-t}(t) \geq F_{a_0}(t) \quad \text{for all} \quad a \in A
\]
Now for \(\lambda \in (0,1)\) we have:
\[
F_{\lambda a}(t) = F_{a-a_0}((1-\lambda)\lambda_0) (t) = F_{a-a_0}((1-\lambda)\lambda_0) (t)
\]
\[
\geq \tau(F_{a-a_0}(1-\lambda)\lambda(t)), \quad F_{a-a_0}(1-\lambda)\lambda(1-\lambda))
\]
\[
\geq \tau(F_{a-a_0}(1-\lambda)\lambda(1-\lambda)) = F_{a-a_0}(t)
\]
So for each \(\lambda \in (0,1), \) we have \(F_{\lambda a}(t) \geq F_{a_0}(t)\) and then \(a_0 \in R_t'(x)\). Hence \(R_t'(x)\) is convex.

Theorem 6: Let \((V, \mathcal{V}, \tau)\) be a Menger PN-space and \(A\) be a subset of \(V\). If \(a_t, a_0 \in R_t'(x)\) and \((1-\lambda)x+\lambda a_0 \in A\), for \(x \in V\) and every scalar \(\lambda \neq 0\), then \((1-\lambda)x+\lambda a_0 \in R_t'(x)\).

Proof: Let \(a_0 \in R_t'(x), t > 0\), and \(x \in V\), then:
\[
F_{a_0}(t) \geq F_{a_0}(t) \quad \text{for all} \quad a \in A
\]
Then for \(\lambda \neq 0\):
\[
F_{\lambda a}(t) = F_{\lambda(1-\lambda)x+(\lambda-a_0)}(t)
\]
\[
= F_{\lambda(1-\lambda)x+(\lambda-a_0)}(t)
\]
\[
\geq \tau(F_{\lambda a}(1-\lambda\lambda(t)), \quad F_{\lambda a}(1-\lambda\lambda(1-\lambda))
\]
\[
\geq \tau(F_{\lambda a}(1-\lambda\lambda(1-\lambda)) = F_{\lambda a}(t)
\]
for all \(a \in A\), thus \((1-\lambda)x+\lambda a_0 \in R_t'(x)\).

Example 7: Let \(V = \mathbb{R}^2\). Define \(\mathcal{V}: \mathbb{R}^2-D^*\) as:
\[
F_{(x_1,x_2)}(t) = (\exp(\sqrt{x_1^2 + x_2^2}/t))^{-1}
\]
Then \((V, \mathcal{V}, \tau)\) is a Menger PN-space where \(\tau(F(t), G(t)) = F(t)G(t)\) for every \(F\) and \(G\) in \(D^*\).

Let \(A = \{(x_1, x_2) \in \mathbb{R}^2 | -1 \leq x_1 \leq 1, 0 \leq x_2 \leq x_1^2\} \) and \(x = (0,2)\). Then for every \(t > 0, (1,1), (-1,1) \in R_t'(0,2)\).

Theorem 8: For \(t > 0\), let A be a P-coproximinal subspace of a PN-space \((V, \mathcal{V})\). Then:
- If \(A\) is a compact set then \(R_t'(x)\) is compact, for every \(x \in V\).
- If \(A\) is a closed set then \(R_t'(x)\) is closed, for every \(x \in V\).

Proof:
- Suppose \(x \in V\) and \(\{a_n\}\) is a sequence in \(R_t'(x)\). Since \(x-a_n \in A\) and \(A\) is a compact set, there is a subsequence \(\{x-a_n\}\) that converges to \(u_0 \in A\).
- Since \(x-u_0 = a_n\) therefore \(a_n \in R_t'(x)\).
- It is similar to (1).

Definition 9: Let \((V, \mathcal{V}, \tau)\) be a Menger PN-space and \(A\) be a subset of \(V\). An element \(x \in A\) is said to be orthogonal to an element \(y \in V\), and we denote \(x \perp y\), if \(F_{x+y}(t) \neq F_x(t)\) for all scalar \(\lambda \in \mathbb{R}, \lambda > 0\) and \(t > 0\).

Also, An element \(x \in V\) is said to be orthogonal to \(A\), and we denote \(x \perp A\), if \(y \perp x\) for all \(y \in A\).

Theorem 10: Let \((V, \mathcal{V}, \tau)\) be a Menger PN-space and \(A\) be a subspace of \(V\). Then for \(x \in V, y_0 \in R_t'(x)\) if and only if \(A \perp x, y_0\) for every \(t > 0\).

Proof: Suppose \(x \in V\) and \(A \perp x, y_0\). Then \(F_{x+y}(t) \neq F_x(t)\) for all \(a \in A\) and scalar \(\lambda \in \mathbb{R}, \lambda \neq 0\) and \(t > 0\), if and only if \(F_{x+y}(t) \neq F_x(t)\) for every \(\lambda \neq 0\), and if only if \(F_{x+y}(t) \neq F_x(t)\) for all \(a \in A\) and for every \(t > 0\), if and only if \(y_0 \in R_t'(x)\).

Remark 11: Let \((V, \mathcal{V}, \tau)\) be a Menger PN-space and \(A\) be a subspace of \(V\):
\[
(R_t')'#(0) = \{x \in V: F_x(t) \neq F_{a} (t) \quad \text{for all} \quad a \in A\}
\]
\[t > 0\} = \{x \in V: A \perp y_0\}
\[
\bar{A} = \{x \in V: A \perp y_0\}
\]

Theorem 12: Let \(A\) be subspace of a Menger PN-space \((V, \mathcal{V}, \tau)\), then \(\bar{A} \cap A = \{0\}\).

Proof: Let \(a \in \bar{A} \cap A\), we show that \(a = 0\). To see this, we have \(a \in \bar{A}\), then \(A \perp a\) and \(a \in A\), this implies that \(h \perp a\) for all \(h \in A\). Therefore, \(F_{h+a}(t) \neq F_h(t)\) for all \(h \in A, t > 0\) and every scalar \(\lambda\). Now, if we choose \(\lambda = -1/3\) and \(h = a\), then \(F_{a+a}(t) \neq F_a(t)\), and so \(F_{3/2}(t) = F_{a+a}(t)\). Hence \(a = 0\), i.e., \(A \cap A = \{0\}\). But \(0 \in \bar{A} \cap A\), together, we get \(A \cap A = \{0\}\).
Theorem 13: For $t > 0$, let $A$ be a P-coproximinal subspace of a Menger PN space $(V, \mathfrak{g}, \tau)$. If $A$ is a convex set, then $A$ is P-coChebyshev, for every $x \in V$.

Proof: Suppose $t > 0, x \in V$ and $a, a \in R'_A(x)$; then $x-a \in A$. Put $\tilde{a} = x-a$, and $\tilde{a} = x-a$ and let us have $x = a + \tilde{a}$. Since $2(\tilde{a} + \tilde{a}) \in A$, it follows that $a + a \in A \Rightarrow \{0\}$; then $a = a$.

Theorem 14: For $t > 0$. Let $(V, \mathfrak{g}, \tau)$ be a Menger PN-space, $A$ and $A$ be subspaces of $V$, such that $A \subseteq A$, and let $x \in V$. Then:

$$R'_A(R'_A(x)) \leq R'_A(x)$$

Proof: Suppose $a \in R'_A(A(x))$, then $a \in R'_A(A)$. Thus, $F_{a+x}(t)$ for all $a \in A$. Now, since $a \in A$,

$$F_{a+x}(t) = F_{a+x}(a + a) \leq F_{a+x}(a) \leq F_{a+x}(t).$$

since $F_{a+x}(t)$ for all $a \in A$, then $A \subseteq A(x)$. Hence $R'_A(x) = A(x)$.

Corollary 15: For $t > 0$. Let $(V, \mathfrak{g}, \tau)$ be a Menger PN-space, and $A$ be subspace of $V$. Then $R'_A(x) = A(x)$. 

Proof: Let $a \in A(x)$. If and only if $\tilde{a} = x-\tilde{a}$, where $\tilde{a} \in A$, and $\tilde{a} = x-\tilde{a}$, if and only if $a \in A$, and $\tilde{a} = x-\tilde{a}$, if and only if $a \in A$. Therefore, $R'_A(x) = A(x)$.

Theorem 16: Let $A$ be subspace of a Menger PN-space $(V, \mathfrak{g}, \tau)$, then:

- $A$ is a P-coproximinal subspace if and only if $V = A + A$.
- $A$ is a P-coChebyshev subspace if and only if $V = A + A$.

Proof: 

(\Rightarrow) Let $t > 0$, $V = A + A \Rightarrow \{a+y; a \in A, y \in A\}$, and $x \in V$. Then $x = a + y$, where $a \in A, y \in A$. Since $y \in A = A - (a+y)$, then $0 \in A(x, y)$. Since $x = a + y$, then $y = x-a$, so $R'_A(y) = R'_A(x-a)$; this implies that $0 \in R'_A(y) = R'_A(x-a)$.

Then $F_{0 \cdot (x-a)}(t) \leq F_{a \cdot (x-a)}(t)$, so $F_{a \cdot (x-a)}(t) = F_{a \cdot (x-a)}(t)$ where $a \in R'_A(x)$. Therefore $A$ is P-coproximinal.

(\Rightarrow) Suppose that $A$ is a P-coChebyshev subspace and $x \in V$, $x = a + \tilde{a}$, where $a, \tilde{a} \in A$. We show that $a_1 = a_2$, and $\tilde{a}_1 = \tilde{a}_2$, since $x = a_1 + \tilde{a}_1 = a_2 + \tilde{a}_2$, then $x-a_1 = \tilde{a}_1$, $x-a_2 = \tilde{a}_2$, this implies that $a_1, a_2 \in R'_A(x)$.

Therefore, $a_1 = a_2$, because $A$ is a P-coChebyshev, it follows that $\tilde{a}_1 = \tilde{a}_2$. Thus $V = A$. 

(\Rightarrow) Let $V = A + A$ and suppose for $x \in V$, there exist $a_1, a_2 \in R'(x)$. We show $a_1 = a_2$. Since $a_1, a_2 \in R'(x)$, then $x-a_1, x-a_2 \in A$. Therefore, $x = a_1 + \tilde{a}_1 = a_2 + \tilde{a}_2$, where $\tilde{a}_1 = x-a_1$, and $\tilde{a}_2 = x-a_2$. Since $V = A + A$, then $a_1 = a_2$ and $\tilde{a}_1 = \tilde{a}_2$. Hence $A$ is P-coChebyshev.

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