Definition 1: Atanassov (1986) Let X be a nonempty set. A set \( A = \{< x, \mu_A(x) \nu_A(x)>| x \in X \} \) is said to be an intuitionistic fuzzy set of \( X \) if mapping \( \mu_A : X \rightarrow [0,1] \) and \( \nu_A : X \rightarrow [0,1] \) satisfy \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \) for all \( x \in X \).

A collection of intuitionistic fuzzy sets of \( X \) is denoted by \( IFS(X) \). Let \( A, B \in IFS(X) \), \( A = \{< x, \mu_A(x) \nu_A(x)>| x \in X \} \) and \( B = \{< x, \mu_B(x) \nu_B(x)>| x \in X \} \). Consider the relation between \( A \) and \( B \) as follow:

\( A \leq B \) if and only if \( \mu_A(x) \leq \mu_B(x) \) and \( \nu_A(x) \geq \nu_B(x) \) for all \( x \in X \).

\( A = B \) if and only if \( \mu_A(x) = \mu_B(x) \) and \( \nu_A(x) = \nu_B(x) \) for all \( x \in X \).

\( A \cap B = \{< x, \min(\mu_A(x), \mu_B(x)) \nu_A(x) \nu_B(x)>| x \in X \} \)

\( A \cup B = \{< x, \max(\mu_A(x), \nu_B(x)) \nu_A(x) \nu_B(x)>| x \in X \} \)

Otherwise,

\( \bigcap_{j \in J} A_j = \{< x, \bigwedge_{j \in J} \mu_{A_j}(x) \bigvee_{j \in J} \nu_{A_j}(x)>| x \in X \} \)

\( \bigcup_{j \in J} A_j = \{< x, \bigvee_{j \in J} \mu_{A_j}(x) \bigwedge_{j \in J} \nu_{A_j}(x)>| x \in X \} \)

\( \square A = \{< x, \mu_A(x) - \nu_A(x)>| x \in X \} \)

\( \lozenge A = \{< x, \nu_A(x) \nu_B(x)>| x \in X \} \)

Definition 2: (Li and Wang, 2000) Let \( f \) be a mapping from the set \( X \) to the set \( Y \) and \( B = \{< x, \mu_B(x), >| x \in X \} \). a intuitionistic fuzzy set of \( Y \). The inverse image of \( B \), denoted by \( f^{-1}(B) \), is the intuitionistic fuzzy set of \( X \) defined by:

\[ f^{-1}(B) = \{< x, \mu_B(f(x)), \nu_B(f(x))>|x \in X\} \]

Conversely, let \( A = \{< x, \mu_A(x) \nu_A(x) >| x \in X \} \) be an intuitionistic fuzzy set of \( X \). Then the image of \( A \), denoted by \( f(A) \), is the intuitionistic fuzzy set of \( Y \) given by:
Definition 1: Let $(X, *, 0)$ be a B-algebra. A intuitionsitic fuzzy B-algebra is a fuzzy B-algebra $(A, *)$ such that for all $x, y, z \in X$:

\[(B1)\] \(x * 0 = x\)

\[(B2)\] \(x * y = x\)

\[(BCH3)\] \((x * y) * z = x * (y * (0 * y))\)

for all \(x, y, z \in X\).

In Young et al. (2002), the concept of fuzzy B-algebras is introduced. A fuzzy set \(\mu\) in B-algebra \(X\) is said to be an anti-fuzzy B-algebra of \(X\) if for all \(x, y \in X\), \(\mu(x * y) = \mu(x) \wedge \mu(y)\).

**INTUITIONSISTIC FUZZY B-ALGEBRAS**

**Proposition 1:** Let \(A = \{< x, \mu_i(x), v_i(x), > | x \in X \}\) be a intuitionsitic fuzzy B-algebra. Then, \(\mu_i(0) \geq \mu_i(x)\) and \(v_i(0) \leq v_i(x)\) for all \(x \in X\).

**Proof:** Since \(x * x = 0\) for all \(x \in X\), the property holds for all \(x \in X\). The completeness of the proof follows.

For any elements \(x\) and \(y\) of \(X\), let us write \(\prod_{x * y}\) for \((x * x) * (x * (0 * y))\) where \(x\) occurs \(n\) times.

**Proposition 2:** Let \(A = \{< x, \mu_i(x), v_i(x), > | x \in X \}\) be a intuitionsitic fuzzy B-algebra and let \(n \in N\). Then for all:

\[\mu_i(\prod_{x * y}) \geq \mu_i(x)\]

\[v_i(\prod_{x * y}) \leq v_i(x)\]

whenever \(n\) is odd.

**Theorem 5:** If a intuitionsitic fuzzy set \(A\) of \(X\) satisfies (IFB1) and (IFB2), then \(A\) is a intuitionsitic fuzzy B-algebra.

**Proof:** Suppose that \(A\) satisfies (IFB1) and (IFB2) and let \(x, y \in X\). Then,

\[v_i(x * y) = v_i(x * (0 * (0 * y)))\]
Proof: For all \( x, y \in X \), we get:
\[
\mu_{A \cap B}(x \ast y) = \mu_A(x \ast y) \wedge \mu_B(x \ast y) \\
\geq [\mu_A(x) \wedge \mu_A(y)] \wedge [\mu_B(x) \wedge \mu_B(y)] \\
= [\mu_A(x) \wedge \mu_B(x)] \wedge [\mu_A(y) \wedge \mu_B(y)] \\
= \mu_{A \cap B}(x) \wedge \mu_{A \cap B}(y)
\]
similarly, we have that:
\[
\nu_{A \cap B}(x \ast y) \leq \nu_{A \cap B}(x) \vee \nu_{A \cap B}(y)
\]
and hence \( A \cap B \) is a intuitionsitic fuzzy B-algebra.

Corollary 7: Let \( A_i \) be intuitionsitic fuzzy B-algebra of \( X \) for all \( i \in I \). Then \( \bigcap_{i \in I} A_i \) is also a intuitionsitic fuzzy B-algebra of \( X \). Similarly, we have:

Theorem 8: Let \( A \) and \( B \) are intuitionsitic fuzzy B-algebras of \( X \). Then \( \square A \) and \( \Diamond A \) are intuitionsitic fuzzy B-algebras of \( X \).

Definition 2: A intuitionsitic fuzzy set \( A \) of \( X \) has sup-inf property if, for any \( T \subseteq X \), there exist \( x_0, y_0 \in T \) such that \( \mu_0(x_0) = \sup_{x \in T} \mu_A(x) \) and \( \nu_0(y_0) = \inf_{y \in T} \nu_A(y) \).

Theorem 9: Let \( f \) be a homomorphism from a B-algebra \( X \) into a B-algebra \( Y \) and \( A \) an intuitionsitic fuzzy B-algebra of \( X \) with sup-inf property. Then the image \( f(A) \) of \( A \) is a intuitionsitic fuzzy B-algebra of \( Y \).

Proof: Let \( A = \{ x, \mu_A(x), \nu_A(x), x \in X \} \) and let \( y_1, y_2 \in Y \). We consider the following cases:

Case (1): If \( f^{-1}(y_1) = \emptyset \) or \( f^{-1}(y_2) = \emptyset \), then \( f^{-1}(y_1 \ast y_2) = \emptyset \). And so \( \hat{f}(\mu_A(y_1 \ast y_2)) = 0 \) and \( \hat{f}(\nu_A(y_1 \ast y_2)) = 1 \). Thus \( \hat{f}(\mu_A(y_1 \ast y_2)) = 0 \) and:
\[
\hat{f}(\nu_A(y_1 \ast y_2)) = 1 \Rightarrow \hat{f}(\nu_A(y_1)) \vee \hat{f}(\nu_A(y_2))
\]

Case (2): If \( f^{-1}(y_1) = \emptyset \) and \( f^{-1}(y_2) = \emptyset \), then let \( x_{10}, x_{20} \in X \) such that \( \mu_A(x_{10}) = \inf_{y \in f^{-1}(y_1)} \mu_A(x) \), \( \mu_A(x_{20}) = \inf_{y \in f^{-1}(y_2)} \mu_A(x) \) and:
\[
\mu_A(x_1 \ast x_2) = \inf_{y \in f^{-1}(y_1 \ast y_2)} \mu_A(x)
\]
Then \( f(\mu_A(y_1 \ast y_2)) = \sup_{y \in f^{-1}(y_1 \ast y_2)} \mu_A(z) \).
Therefore, \( f(A) \) is a intuitionsitic fuzzy B-algebra of \( Y \).

Theorem 10: Let \( f \) be a homomorphism from a B-algebra \( X \) onto a B-algebra \( Y \) and \( A \) an intuitionsitic fuzzy B-algebra of \( Y \). Then the preimage \( f^{-1}(A) \) of \( A \) is a intuitionsitic fuzzy B-algebra of \( X \).

Proof: Let \( x, y \in X \). Since \( A \) is a intuitionsitic fuzzy B-algebra of \( Y \), we have that:
\[
\mu_{f^{-1}(A)}(x \ast y) = \mu_A(f(x \ast y)) \\
= \mu_A(f(x) \ast f(y)) \\
\geq \mu_A(f(x)) \wedge \mu_A(f(y)) \\
= \mu_{f^{-1}(A)}(x) \wedge \mu_{f^{-1}(A)}(y)
\]
Similarly,
\[
\nu_{f^{-1}(A)}(x \ast y) \leq \nu_{f^{-1}(A)}(x) \vee \nu_{f^{-1}(A)}(y)
\]
Hence \( f^{-1}(A) \) is a intuitionsitic fuzzy B-algebra of \( X \).

Corollary 11: Let \( f \) be a homomorphism from a B-algebra \( X \) onto a B-algebra \( Y \). Then the following conclusions hold:

- If for all \( j \in J, A_j \) are intuitionsitic fuzzy B-algebras of \( X \), then \( f(\bigcap_{j \in J} A_j) \) is intuitionsitic fuzzy B-algebra of \( Y \).
- If for all \( t \in T, B_t \) are intuitionsitic fuzzy B-algebras of \( Y \), then \( f^{-1}(\bigcap_{t \in T} B_t) \) is intuitionsitic fuzzy B-algebras of \( X \).
Theorem 12: Let $f$ be an isomorphism from a B-algebra $X$ onto a B-algebra $Y$. If $A$ is a intuitionsitic fuzzy B-algebra of $X$, then $f^{-1}(f(A)) = A$.

Proof: For any $x \in X$, let $f(x) = y$, since $f$ is an isomorphism, $f^{-1}(y) = \{x\}$. Thus $f^{-1}(f(x)) = f(\mu_A(x) = f(\mu_A(y) = \bigwedge_{x \in f^{-1}(y)} \mu_A(x) = \mu_A(x)$ and:

$$f^{-1}(f_A(x)) = f_A(x) = f_A(y)$$

therefore $f^{-1}(f(A)) = (A)$.

Corollary 13: Let $f$ be an isomorphism from a B-algebra $X$ onto a B-algebra $Y$. If $B$ is a intuitionsitic fuzzy B-algebra of $Y$, then $f^{-1}(f(B)) = (B)$.

Corollary 14: Let $f: X \to X$ be an automorphism. If $A$ is a intuitionsitic fuzzy B-algebra of $X$, then:

$$f(A) = A \iff f^{-1}(A) = A$$

A intuitionsitic fuzzy set: $R = \{(x, y), \mu_R(x, y), \nu_R(x, y)\} | x \in X, y \in Y \} \in \text{IFS}(X \times Y)$ is called a binary intuitionsitic fuzzy relation (Lei et al., 2005) from $X$ into $Y$. A binary intuitionsitic fuzzy relation from $X$ into $Y$ is said to be a binary intuitionsitic fuzzy relation on $X$ if $X = Y$.

Definition 3: Let $A = \{(x, x), \mu_A(x), \nu_A(x)\} | x \in X \} \in \text{IFS}(X)$ A binary intuitionsitic fuzzy relation:

$$R = \{(x, y), \mu_R(x, y), \nu_R(x, y)\} | x, y \in X \}$$

on $X$ is called a intuitionsitic fuzzy relation on $A$ if $\mu_R(x, y) = \mu_A(x) \wedge \mu_A(y)$ and $\nu_R(x, y) = \nu_A(x) \vee \nu_A(y)$ for all $x, y \in X$.

Lemma 15: Let $A = \{(x, x), \mu_A(x), \nu_A(x)\} | x \in X \} \in \text{IFS}(X)$ and $B = \{(x, x), \mu_B(x), \nu_B(x)\} | x \in X \} \in \text{IFS}(X)$ be intuitionsitic fuzzy sets of $X$. A Cartesian product of $A$ and $B$ defined by:

$$A \times B = \{(x, y), \mu_{A \times B}(x, y), \nu_{A \times B}(x, y)\} | x, y \in X \}$$

where $\mu_{A \times B} = \mu_A(x) \wedge \mu_B(y)$, $\nu_{A \times B} = \nu_A(x) \vee \nu_B(y)$. Then $A \times B$ is a binary intuitionsitic fuzzy relation on $X$.

Theorem 16: Let $A = \{(x, x), \mu_A(x), \nu_A(x)\} | x \in X \} \in \text{IFS}(X)$ and $B = \{(x, x), \mu_B(x), \nu_B(x)\} | x \in X \} \in \text{IFS}(X)$ be intuitionsitic fuzzy B-algebras of $X$. Then $A \times B$ is a intuitionsitic fuzzy B-algebra of $X \times X$.

Proof: Since $A, B$ are intuitionsitic fuzzy B-algebras of $X$, we have:

$$\mu_{A \times B}(x, y) = \mu_A(x) \wedge \mu_B(y)$$

$$\nu_{A \times B}(x, y) = \nu_A(x) \vee \nu_B(y)$$

for all $(x, y), (x', y') \in X \times X$.

Similarly,

$$\nu_{A \times B}((x, y) \wedge (x', y')) = \nu_{A \times B}(x, y) \vee \nu_{A \times B}(x', y')$$

for all $(x, y), (x', y') \in X \times X$.

Hence $A \times B$ is a intuitionsitic fuzzy B-algebra of $X \times X$.

Theorem 17: Let $A = \{(x, x), \mu_A(x), \nu_A(x)\} | x \in X \} \in \text{IFS}(X)$ and $B = \{(x, x), \mu_B(x), \nu_B(x)\} | x \in X \} \in \text{IFS}(X)$ be intuitionsitic fuzzy sets of a B-algebra $X$ such that $A \times B$ is a intuitionsitic fuzzy B-algebra of $X \times X$.

- Either $\mu_A(x) \leq \mu_A(0)$ or $\mu_B(x) \leq \mu_B(0)$ for all $x \in X$.
- Either $\nu_A(x) \leq \nu_A(0)$ or $\nu_B(x) \leq \nu_B(0)$ for all $x \in X$.
- If $\mu_A(x) \leq \mu_A(0)$ for all $x \in X$, then $\mu_A(x) = \mu_A(0)$ or $\mu_B(x) = \mu_B(0)$.
- If $\nu_A(x) \leq \nu_A(0)$ for all $x \in X$, then $\nu_A(x) = \nu_A(0)$ or $\nu_B(x) = \nu_B(0)$.
- If $\mu_A(x) \leq \mu_B(x) \leq \mu_B(0)$ for all $x \in X$, then $\mu_A(x) = \mu_B(x) = \mu_B(0)$.
- If $\nu_A(x) \leq \nu_B(x) \leq \nu_B(0)$ for all $x \in X$, then $\nu_A(x) = \nu_B(x) = \nu_B(0)$.
- Either $\mu_A$ or $\mu_B$ is a fuzzy B-algebra (Young et al., 2002) of $X$.
- Either $\nu_A$ or $\nu_B$ is an anti-fuzzy B-algebra of $X$.
Proof: (i) Suppose that \( \mu_A(x) \leq \mu_A(0) \) and \( \mu_B(y) > \mu_B(0) \) for some \( x, y \in X \). Then:
\[
\mu_{\text{iFBA}}(x, y) = \mu_A(x) \wedge \mu_B(y) > \mu_A(0) \wedge \mu_B(0) = \mu_{\text{iFBA}}(0,0).
\]
This is a contradiction. Hence (i) holds.

(ii) is by similar method to part (i).

(iii) Assume that there exist \( x, y \in X \) such that \( \mu_A(x) > \mu_A(0) \) and \( \mu_B(y) > \mu_B(0) \). Then
\[
\mu_{\text{iFBA}}(x, y) = \mu_A(x) \wedge \mu_B(y) > \mu_A(0) \wedge \mu_B(0) = \mu_{\text{iFBA}}(0,0),
\]
which is a contradiction. Hence (iii) holds.

(iv) (v) and (vi) are by similar method to part (iii).

(vii) Since by (i) either \( \mu_A(x) \leq \mu_A(0) \) or \( \mu_B(y) > \mu_B(0) \) for all \( x \in X \), without loss of generality we may assume that \( \mu_A(x) \leq \mu_A(0) \) for all \( x \in X \). From (v), it follows that \( \mu_A(x) < \mu_A(0) \) or \( \mu_B(y) > \mu_B(0) \). If \( \mu_B(y) > \mu_B(0) \), then \( \mu_{\text{iFBA}}(0, x) = \mu_A(0) \wedge \mu_B(x) = \mu_B(x) \). Let \( x_1, x_2, (y_1, y_2) \in X \times X \). Since \( X \times B \) is a intuitionistic fuzzy B-algebra of \( X \times X \), we have:
\[
\mu_{\text{iFBA}}(x_1, x_2) \geq \mu_{\text{iFBA}}(y_1, y_2)
\]
(IFO3)
\[
= [\mu_A(x_1) \wedge \mu_B(x_2)] \wedge [\mu_A(y_1) \wedge \mu_B(y_2)]
\]
If we take \( x_1 = y_2 = 0 \), then
\[
\mu_B(x_1, y_2) = \mu_B(0, x_2) \wedge [\mu_A(0) \wedge \mu_B(0)]
\]
\[
= \mu_B(x_2, y_2)
\]
This proves that \( \mu_B \) is a fuzzy B-algebra of \( X \). Now we consider the case \( \mu_B(x) < \mu_B(0) \) for all \( x \in X \). Suppose that \( \mu_B(y) > \mu_B(0) \) for some \( y \in X \). Then \( \mu_A(0) < \mu_B(y) \leq \mu_B(0) \). Since \( \mu_A(x) \leq \mu_A(0) \) for all \( x \in X \), we have \( \mu_B(0) \geq \mu_B(x) \) for all \( x \in X \). Hence \( \mu_{\text{iFBA}}(0, x) = \mu_A(x) \wedge \mu_B(0) = \mu_A(x) \). Taking \( x_1, x_2 = 0 \) in (IFO3), then:
\[
\mu_{\text{iFBA}}(x_1, y_1) = \mu_{\text{iFBA}}(x_2, y_1)
\]
\[
\mu_{\text{iFBA}}(0, x_1) \geq [\mu_A(x_1) \wedge \mu_B(0)] \wedge [\mu_A(y_1) \wedge \mu_B(0)]
\]
\[
= \mu_A(x_1) \wedge \mu_B(y_1)
\]
Therefore \( \mu_A \) is a fuzzy B-algebra of \( X \).

(viii) is by similar method to part (vii) and the proof is completed.

From the proof of Theorem 17 (vii) and Theorem 17 (viii), the following results hold up.

**Theorem 18:** Let \( A = \{x, \mu_A(x), v_A(x), x \in X\} \) and \( B = \{x, \mu_B(x), v_B(x), x \in X\} \) be intuitionistic fuzzy sets of a B-algebra \( X \) such that \( A \times B \) is a intuitionistic fuzzy B-algebra of \( X \times X \). Then:
- If \( \mu_A(x) \leq \mu_A(0) \wedge \mu_B(0) \) and \( v_A(x) \leq v_A(0) \wedge \mu_B(0) \) for all \( x \in X \), then \( A \) is a intuitionistic fuzzy B-algebra of \( X \).
- If \( \mu_A(x) \leq \mu_A(0) \wedge \mu_B(0) \) and \( v_A(x) \leq v_A(0) \wedge \mu_B(0) \) for all \( x \in X \), then \( B \) is a intuitionistic fuzzy B-algebra of \( X \).

**Definition 4:** Let \( A = \{x, \mu_A(x), v_A(x), x \in X\} \in \text{IFS}(X) \). A intuitionistic fuzzy relation:
\[
R = \{(x, y), v_A(x, y), v_B(x, y), x, y \in X\}
\]
on \( X \) is called a strongest intuitionistic fuzzy relation on \( A \) if:
\[
\mu_B(x, y) = \mu_A(x) \wedge \mu_B(y) \quad \text{and} \quad v_B(x, y) = v_A(x) \wedge v_B(y)
\]
for all \( x, y \in X \).

**Proposition 19:** For a given intuitionistic fuzzy set \( A = \{x, \mu_A(x), v_A(x), x \in X\} \) of a B-algebra \( X \), let \( R \) be a strongest intuitionistic fuzzy relation on \( A \). If \( R \) is an intuitionistic fuzzy B-algebra of \( X \times X \), then \( \mu_A(x) \geq \mu_A(0) \) and \( v_A(x) \geq v_A(0) \) for all \( x \in X \).

**Proof:** Since \( R \) is an intuitionistic fuzzy B-algebra, we have \( \mu_B(x, x) \leq \mu_B(0,0) \) and \( v_B(x, x) \geq v_B(0,0) \) for all \( x \in X \), that is, \( \mu_B(x, x) \wedge v_B(x, x) \geq v_B(0,0) \). So \( \mu_A(x) \geq \mu_A(0) \) and \( v_A(x) \geq v_A(0) \) for all \( x \in X \).

**Theorem 20:** Let \( A = \{x, \mu_A(x), v_A(x), x \in X\} \) Let be a intuitionistic fuzzy set of B-algebra \( X \) and:
\[
R = \{(x, y), v_A(x, y), v_B(x, y), x, y \in X\}
\]
strongest intuitionistic fuzzy relation on \( A \). Then \( A \) is an intuitionistic fuzzy B-algebra of \( X \) if and only if \( R \) is an intuitionistic fuzzy B-algebra of \( X \times X \).

**Proof:** Assume that \( A \) is an intuitionistic fuzzy B-algebra of \( X \). Then:
\[
\mu_B(x_1, x_2) \leq \mu_B(x_1, y_1) \wedge \mu_B(x_2, y_2)
\]
\[
\mu_B((x_1, x_2) \ast (y_1, y_2)) = \mu_B(x_1 \ast y_1, x_2 \ast y_2)
\]
\[
\begin{align*}
  &= \mu_a(x_1^*y_1) \land \mu_a(x_2^*y_2) \\
  \geq [\mu_a(x_1) \land \mu_a(y_1)] \land [\mu_a(x_2) \land \mu_a(y_2)] \\
  &= [\mu_a(x_1) \land \mu_a(x_2)] \land [\mu_a(y_1) \land \mu_a(y_2)] \\
  &= \mu_b(x_1, x_2) \land \mu_b(y_1, y_2)
\end{align*}
\]

for all \((x_1, x_2), (y_1, y_2) \in X \times X\). Similarly, we have that \(\nu_{A'}(x_1, x_2, *(y_1, y_2)) \leq \nu_{A'}(x_1, x_2) \lor \nu_{A'}(y_1, y_2)\) for all \((x_1, x_2), (y_1, y_2) \in X \times X\). Hence \(R\) is an intuitionistic fuzzy B-algebra of \(X \times X\).

Conversely, suppose that \(R\) is an intuitionistic fuzzy B-algebra of \(X \times X\) and let \(x, y \in X\), then,

\[
\begin{align*}
  \mu_A(x^*y) &= \mu_{A'}(x^*y, x^*y) \\
  &= \mu_{A'}((x, x)^*, (y, y)) \\
  \geq \mu_{B'}(x, x)^* \land \mu_{B'}(y, y) \\
  &= \mu_{B'}(x) \land \mu_{B'}(y)
\end{align*}
\]

Similarly, we have \(\nu_{A'}(x^*y) \leq \nu_{A'}(x) \lor \nu_{A'}(y)\). And completes the proof.

REFERENCES


