Synchronization of Chaotic Systems by Using Limited Nussbaum Gain Method

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Abstract: In this study, we propose a novel kind of limited Nussbaum gain method to solve the unknown control direction problem of chaotic systems. It is necessary to consider the unknown control direction situation when chaos synchronization is used for secure communication. Also it is very useful to design a small gain with a Nussbaum method if the Nussbaum gain method is used in a real engineering system. The smaller a gain is designed, the safer and more stable a Nussbaum gain method is used. According to the above principles, a limited gain Nussbaum method is used to solve the uncertainties of chaotic system, such as unknown parameters, unknown nonlinear functions and unknown input or control directions. At last, detailed numerical simulation is done to testify the rightness and effectiveness of the proposed method.

Keywords: Chaos, limited gain, nussbaum gain, synchronization, unknown control direction

INTRODUCTION

Chaos systems have complex dynamical behaviors that possess some special features such as being extremely sensitive to tiny variations of initial conditions and having bounded trajectories with a positive leading Lyapunov exponent and so on Hu et al. (2007), Gao et al. (2007), Elabbasy et al. (2006) Tang and Wang (2005) Ge and Yang (2007), Gauthier et al. (1992), Khalil and Saberi (1987) and Hao and Wei (2006). Synchronization of chaos systems with unknown parameters was investigated widely by researchers from various fields. There are also many papers researched others kinds of uncertainties such as input uncertainty, unknown functions and unmolded dynamics. It is also necessary to research those uncertain situation above since they are very useful if the synchronization of chaos is used for secure communication.

There are also many problems should be considered if a real communication system is constructed with chaotic synchronization theory (Hao and Wei, 2007; Wang-Long and Kuo-Ming, 2007). The unknown control direction problem is one of those problems should be researched and it is also a very meaningful research since this problem is very difficult not only in theory but also in engineering. In fact, we have to solve it firstly in theory. In the future, maybe it can be used in engineering. But it is a great challenge and it is real exist in practice. To solve this problem, Nussbaum proposed a method which was named by his name later. It is a effective method for some simple system at first, so the Nussbaum gain method is mostly used in some low order systems.

There is another problem should be considered when the Nussbaum gain method is used to design a real system and is used in engineering practice. It is that the gain of the system should be as small as possible that it because the energy of real system is limit and also if the gain can be designed to be smaller, the system is comparatively more stable. So for Nussbaum gain method, the gain should be limit at least. Then it can be defined as limit gain problem for Nussbaum gain method.

It is obvious that it is very useful to design a limit Nussbaum gain method for the synchronization of uncertain chaotic system for secure communication. But it is also very difficult to solve the limit gain problem. In this study, a new kind of novel Nussbaum gain method is proposed under the above background.

METHODOLOGY

Problem description: Take the universe chaos system for an example, the driven system can be described as follow:

\[
\dot{x} = f_x(x) + F_x(x)\theta + \Delta(x,t)
\]

(1)

The response system can be written as:

\[
\dot{y} = f_y(y) + bu
\]

(2)

If it is a three dimension chaotic system, it can be expanded as:

\[
\dot{x}_1 = f_1(x_1, \ldots, x_4) + \sum_{j=1}^{4} F_{1j}(x_1, \ldots, x_4)\theta_{1j} + \sum_{j=1}^{4} \Delta_{1j}(x,t)
\]

(3)

\[
\dot{x}_2 = f_2(x_1, \ldots, x_4) + \sum_{j=1}^{4} F_{2j}(x_1, \ldots, x_4)\theta_{2j} + \sum_{j=1}^{4} \Delta_{2j}(x,t)
\]

(4)
\[ \dot{x}_i = f_{ij}(x_1, \ldots, x_4) \]
\[ + \sum_{j=1}^{q_i} F_{ij}(x_1, \ldots, x_4) \theta_{ij} + \sum_{j=1}^{q_i} \Delta_{ij}(x, t) \]  
(5)

And the slave system can be written as:
\[ \dot{y}_i = f_{ij}(y_1, \ldots, y_4) + b_i u_i \]
(6)
\[ \dot{y}_2 = f_{ij}(y_1, \ldots, y_4) + b_2 u_2 \]
(7)
\[ \dot{y}_3 = f_{ij}(y_1, \ldots, y_4) + b_3 u_3 \]
(8)

where, \( \theta_{ij} \) are unknown parameters and \( \Delta_{ij}(x, t) \) are unknown functions and \( b_i \) are unknown input coefficients.

The control objective is to design a synchronization law \( u = u(x, y) \) such that the synchronization can be fulfilled then it has \( \dot{y} = x \).

This study proposes a novel kind of limited Nussbaum gain method to solve the unknown control direction problem of chaotic systems. According to the above simulation results, it is easy to make a conclusion that the proposed limited Nussbaum gain method is effective for the unknown control direction situation of chaotic system synchronizations. Moreover, the limit gain strategy is very useful if a Nussbaum gain method is applied in a real engineering system.

**Assumptions:** To make the below context more easy to understand, it is necessary to make two assumptions for the chaotic system.

**Assumption 1:** The response system has the same dimension as the driven system.

**Assumption 2:** There exists a positive constant \( d_{ij} \) such that the nonlinearities of the driven system satisfies:
\[ f_{ij}(y_1, \ldots, y_4) - f_{ij}(x_1, \ldots, x_4) \]
\[ - \sum_{j=1}^{q_i} F_{ij}(x_1, \ldots, x_4) \theta_{ij} - \sum_{j=1}^{q_i} \Delta_{ij}(x, t) \]
\[ \leq d_{ij} + d_{ij} |S_i| \]

where, \( S_i \) is sliding mode and it is defined as bellows.

Since the driven system is a chaotic system and chaotic systems are bounded, so it is easy to be satisfied by many common chaotic systems.

**Design of nonlinear sliding mode:** Define a new variable as \( z_i = y_i - x_i \), the error system can be written as:
\[ \dot{z}_i = f_{ij}(y_1, \ldots, y_4) - f_{ij}(x_1, \ldots, x_4) \]
\[ - \sum_{j=1}^{q_i} F_{ij}(x_1, \ldots, x_4) \theta_{ij} - \sum_{j=1}^{q_i} \Delta_{ij}(x, t) + b_i u_i \]
(9)

Design a nonlinear sliding mode as:
\[ S_i = z_i \left[ \sum_{i=1}^{n} w_{ij} \left( \int z_i dt \right)^2 \right] \]
(10)
\[ + \left( \sum_{i=1}^{n} w_{ij} \int z_i dt \right) \int z_i dt \]
where, \( w_{ij} > 0 \).

Without considering the input uncertainty, assume the situation that \( b_i = 1 \), then there exists a idea nonlinear sliding mode adaptive control as:
\[ u_{ip} = b_i^{-1} \left( u_{ic} + u_{id} \right) \]
(11)

where,
\[ u_{ic} = - \left( z_i \int w_{ij} \int z_i dt \right) \]
\[ + \left( \sum_{i=1}^{n} w_{ij} \int z_i dt \right) \int z_i dt \]
\[ 2w_{ij} \frac{\int z_i dt \int z_i dt}{w_{ij} + \sum_{i=1}^{n} \left( \int z_i dt \right)^2} \]
(12)

And \( u_{id} \) is designed as:
\[ u_{id} = k_i S_i + \hat{d}_{ij} \text{sgn}(S_i) - \hat{d}_{ij} \frac{S_i}{w_{ij} + \sum_{i=1}^{n} \left( \int z_i dt \right)^2} \]
(13)

define,
\[ \hat{d}_{ij} = d_{ij} - \hat{d}_{ij} \]
(14)

then,
\[ \hat{d}_{ij} = - \hat{d}_{ij} \]
(15)

Define the turning law as:
\[ \dot{S}_i = |S_i| \left| \hat{d}_{ij} \right|^2 \]
(16)

Choose a Lyapunov function as:
\[ V_i = \frac{1}{2} \left( S_i^2 + \hat{d}_{ij}^2 + \hat{d}_{ij}^2 \right) \]
(17)
According to the assumption, it has:
\[ S_i S_j \leq \tilde{d}_{ij} |S_i| + \tilde{d}_{ij} |S_j| - k_{ij} |S_i|^2 \]  \hspace{1cm} (18)

It is easy to prove that:
\[ \dot{V}_i \leq 0 \]  \hspace{1cm} (19)

**Limit nussbaum gain strategy**: Considering the situation that the control direction is unknown and the Nussbaum gain control strategy can be designed as:
\[ u_i = -N(k_i)u_p \]  \hspace{1cm} (20)

And the Nussbaum turning law can be designed as:
\[ \dot{l}_i = k_i z_i u_i \]  \hspace{1cm} (21)

While design the limit gain function as:
\[ k_i = f_z(l_i) \]  \hspace{1cm} (22)

Where \( f_z(l) \) can be chosen as a triangle function.

Choose the Lyapunov function:
\[ V_i = \frac{1}{2}(1 + k_i)z_i^2 + \sum_{j=2}^n \frac{1}{2}(\dot{d}_j)^2 + \frac{1}{2} k_{ij} (\int z_i dt)^2 \]  \hspace{1cm} (23)

It is easy to solve the derivative of the Lyapunov function and get:
\[ \dot{V}_i \leq -\frac{1}{k_i} (b_i N(k_i) + 1) \dot{l_i} \]  \hspace{1cm} (24)

Use the same method as the traditional Nussbaum strategy, it is easy to prove that \( k_i \) is bounded and \( S_i \rightarrow 0 \), so the synchronization of the system can be fulfilled.

**NUMERICAL SIMULATION**

Take the below three dimension chaotic system as an example to do a numerical simulation and the driven system can be written as:
\[ \dot{x}_1 = a(x_2 - x_1) + k_{ib} x_3 \cos x_2 \]
\[ \dot{x}_2 = b x_1 + c x_2 - x_1 x_3 + k_{ib} x_3 \cos x_2 \]
\[ \dot{x}_3 = x_2^2 - h x_3 + k_{ib} (1 + \sin(x_2 x_3)) x_2 \]

Choose \( a = 20, b = 14, c = 10.6 = 2.8, k_{ib} = 0 \), there exists a chaotic attractors. \( a, b, c, h \) are unknown parameters and \( k_{ib} \) is coefficients of uncertain nonlinear functions.

Assume that the structure of the response system is known and its structure is as follows:
\[ \dot{y}_1 = a_j (y_2 - y_1) + b_j u_1 \]
\[ \dot{y}_2 = b_j y_1 - k_j y_1 y_3 + u_2 \]
\[ \dot{y}_3 = -c_j y_3 + h_j y_3^2 + u_3 \]

Choose parameters as \((a_j, b_j, c_j, k_j, h_j) = (10.4, 2.5, 1, 4)\)

And set the initial state of response system as \((y_1, y_2, y_3) = (1, -1, 2)\)

And assume the unknown control direction as:
\[ b_j = \begin{cases} 
1 & 0 < t < 2 \\
-1 & t > 2
\end{cases} \]

The simulation results can be show as follows:
Figure 1 shows that the state \( x_1 \) of driven system can synchronize with the state \( y_1 \) of response system. Figure 2 shows that the state \( x_2 \) of driven system can
synchronize with the state $y_2$ of response system with the interrupt of input direction changed at 2s. Figure 3 shows that the state $x_3$ of driven system can synchronize with the state $y_3$ of response system with the interrupt of input direction changed at 2s. Figure 4 shows that the synchronization error $z_1$ can converged to zero with the...
change of control direction at 2s. Figure 5 shows that the synchronization error $z_2$ can converged to zero with the change of control direction at 2s. Figure 6 shows that the synchronization error $z_3$ can converged to zero with the change of control direction at 2s.

Choose the coefficients of unknown nonlinear function as $k_{th} = 0.2$, the simulation results are show as follows: Figure 7 shows that the state $x_1$ of driven system can synchronize with the state $y_1$ of response system. Figure 8 shows that the state $x_2$ of driven system can synchronize with the state $y_2$ of response system with the interrupt of input direction changed at 2s. Figure 9 shows that the state $x_3$ of driven system can synchronize with the state $y_3$ of response system with the interrupt of input direction changed at 2s. Figure 10 shows that the synchronization error $z_1$ can converged to zero with the change of control direction at 2s. Figure 11 shows that the synchronization error $z_2$ can converged to zero with the change of control direction at 2s. Figure 12 shows that the synchronization error $z_3$ can converged to zero with the change of control direction at 2s. It can make a conclusion that the synchronization can be achieved successfully no matter there are unknown nonlinear functions or not.

CONCLUSION

According to the above simulation results, it is easy to make a conclusion that the proposed limited Nussbaum gain method is effective for the unknown control direction situation of chaotic system synchronizations. What is worth pointing out is that the limit gain strategy is very useful if a Nussbaum gain method is applied in a real engineering system.

REFERENCES


