Current Research Journal of Economic Theory 2(2): 32-40, 2010
ISSN: 2042-485X

Submitted Date: December 07, 2009 Accepted Date: December 26, 2009 Published Date: March 10, 2010

Limit Cycle, Trophic Function and the Dynamics of Intersectoral Interaction

[1]Gappar Nasritdinov and [2]Ravshanbek Dalimov
[1]Department of Mechanics-Mathematics,
[2]Department of Economics, National University of Uzbekistan,
Universitetskaya Street, Tashkent, Uzbekistan 100174

Abstract: An article develops an idea of implementing the trophic functions in Volterra's "predator-prey" model to the linked intersectoral dynamics of the outputs. The concept of trophic functions and limit cycles are used as key factors in defining the parameters of stable economic dynamics. Two trophic functions for "cars-rolled steel" and "cars-oil products" were built in the article. These empirically based trophic functions were analytically reviewed and constructed in the article providing a tool in the analyses and forecasting of the linked dynamics of the parameters under consideration.

Key words: Attractor, intersectoral consumption dynamics, output, limit cycle, pairs of sectors, trophic function

INTRODUCTION

Inter-sectoral interaction in economics stands for an exchange of the goods between the sectors of economy. The dynamics of intersectoral interaction attempts to describe the mechanism of this exchange in time both to understand what is going on in economy as well as to make feasible forecasts and manage production in the sectors involved in consideration.

The dynamics of intersectoral interaction in the past was mainly described using a concept of "input-output" matrix and a theory of matrix which enabled to compute the links between the outputs of different sectors (Leontieff, 1986). A key disadvantage of this approach is an absence of dynamic elements in the analysis. One of the methods to resolve this issue was provided within an optimal analysis using a theory of the main stream (von Neumann, 1945-1946), which seems quite complicated for practical implementation at the same time.

This article develops non-linear approach towards the dynamics of intersectoral interaction based on a tool of trophic functions and generalized Volterra's "predator-prey" model. Essentially, it is an attempt to use this interdisciplinary non-linear method significantly advanced in biochemistry, mathematical physics and chemical kinetics, towards economic dynamics.

Importance of the trophic functions in analyses of economic dynamics becomes clear if to represent linked regional dynamics of the sectors' outputs as multidimensional Volterra's "predator-prey" system (Dalimov, 2008):

$$Q_k = \sum_{l=1}^{n} \alpha_{lk} V_{kl}(Q_k) Q_l \quad (1)$$

where $Q_k \equiv \frac{dQ}{dt}; \alpha_{kk}$ – constant coefficients, $V_{kk}(Q_k) \equiv 1; V_{kl}(Q_k)$ - trophic functions of the pairs of (l; k) sectors, $l \in [l; n]$; n-number of sectors in economy. One may see that trophic functions influence the dynamics of each sector in economy.

Feasibility of Volterra's "predator-prey" model towards economic dynamics was proved after noticing a similarity of temporal dynamics of linked economic parameters (Dalimov, 2008) to the dynamics of lynxes and rabbits in classic "predator-prey" model (Lotka, 1925; Volterra, 1931) (Fig. 1).

For instance, this way behaves a pair "price-output" of an oil sector as well as of industries having global demand and price structure set in circumstances of perfect competition (free global market). Remarkably, price and output are components of entities / sectors income - a key factor influencing an economic dynamics. The same type of interaction is observed in the linked dynamics of currency pairs "Euro – USD", "USD – JPY", "USD-GBP" etc. For instance, it may be seen on FOREX platforms using 1 and 4 hour intervals.

Trophic functions are defined by the following generalized Volterra's "predator-prey" model:

Corresponding Author: Ravshanbek Dalimov, Department of Economics, National University of Uzbekistan, Universitetskaya Street, Tashkent, Uzbekistan 100174

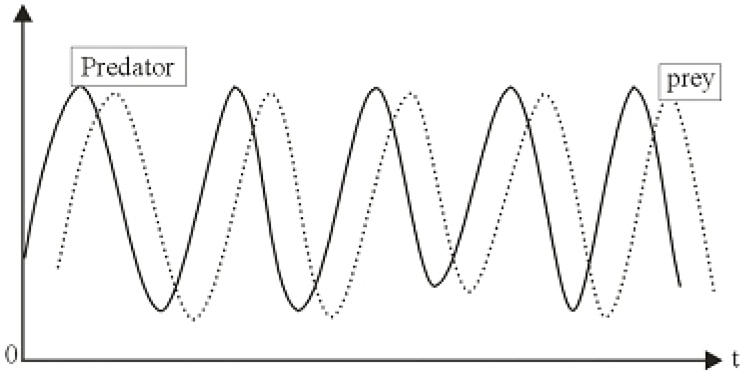

Fig. 1: Temporary dynamics of linked variables

$$\begin{cases} \dot{X} = \alpha X - \beta V(X)Z \\ \dot{Z} = kV(X)Z - mZ \end{cases} \qquad (2)$$

where Z; X are temporarily changing parameters standing for predator and prey respectively;

$$\dot{X} \equiv \frac{dX}{dt}; \quad \dot{Z} \equiv \frac{dZ}{dt};$$

α and β are regenerating and consuming coefficients of the parameter X; with k and m as regenerating and consuming coefficients of the parameter Z.

One may see that when trophic function $V(X) = 0$, parameters X and Z behave exponentially: one is growing according to $Z = Z_0 e^{-mt}; Z_0 - const$. and the other is diminishing according to. So the trophic function $V(X)$ stands as a link for the interacting dynamics of the parameters X and Z, which is one of the main reasons of scholarly and interdisciplinary interest to the trophic functions.

Despite all the success achieved in various sciences in this research, one may clearly state that the work in that direction just begins, as there have not been found preset analytical dependencies yet regarding the type of the trophic function and respective dynamics of the linked parameters observed in practice. What we clearly know at the moment is that the trophic function $V(X)$ is just some kind of a function, while one must have its explicit analytical representation for its analysis and forecasting. This article partially solves this issue.

We consider linear case of the trophic function when $V(X) = X$.

MATERIALS AND METHODS

Main material for the research was the generalized Volterra's "predator-prey" model and trophic function concept. The study was conducted within the premises of Business Administration Unit at Economics Department of the National University of Uzbekistan in 2009. The methods implemented in the article, include analysis of ordinary differential equations and non-linear dynamics of limit cycles. The research was funded by the National grant of the Republic of Uzbekistan OT-F7-082 "Modelling International Economic Integration".

Unstable cycles in Volterra's "predator-prey" model: Let Q_1 be the output of the supply sector, and Q_2 be the output of the processing sector. Consider the model identical to the linear Volterra's "predator-prey" model (Lotka, 1925; Volterra, 1931):

$$\begin{cases} \dfrac{dQ_1}{dt} = \alpha Q_1 - \beta Q_1 Q_2; \\ \dfrac{dQ_2}{dt} = kQ_1 Q_2 - mQ_2; \end{cases} \qquad (3)$$

where $\alpha \rhd 0; \beta \rhd 0; k \rhd 0,$ t stands for time, $\frac{dQ_i}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta Q_i}{\Delta t}; i = 1,2$ - temporal derivative of the output of the sector .

System of equations (3) describes the dynamics of the outputs of sectors ($Q_1 : Q_2$) such as production of steel and cars; wheat and bakery; oil products and cars etc. From the point of microeconomic regulation it is important to find the parameters managing a stable demand of the production in the sector.

First term in the right hand side of the Eq. (1) in the system (3) is regenerating one for the output of the supply sector while the negative second term shows relative consumption of the product of the supply sector. In the 2-d equation the first term of the right hand side shows consumption of the product of the supply sector by the processing sector, while the second term – on natural diminishing of the output in the processing sector caused by amortization, and defects during production.

System of Eq. (3) is fairly known and has variety of modifications in a number of interdisciplinary and natural sciences (Bulmer, 1976; Freedman and Kuang, 1990; Gakkhar *et al*., 2007; Huang, 1990; Rai *et al*., 2007), while in economics there are very few of them (Milovanov, 2001; Zhang, 1991). It is known that one of the points of the equilibrium of the system of Eq. (3) is a point Q^0 (Svirejev and Logofet, 1978):

$$Q^0 = \left(Q_1^0; Q_2^0\right) = \left(\frac{m}{k\beta}; \frac{\alpha}{\beta}\right) \tag{4}$$

It was proved that the system (2) has the following integral:

$$\left[\frac{e^{Q_1^*}}{Q_1^*}\right] \cdot \left[\frac{e^{Q_2^*}}{Q_2^*}\right] = C; \tag{5}$$

where $Q_1^* = \dfrac{Q_1}{Q_1^0}; Q_2^* = \dfrac{Q_2}{Q_2^0}$

Relationship (5) describes a set of the cycles inserted one to the other (Fig. 2) corresponding to phase trajectories of periodic solutions of the system (3). It is known that they are unstable which means that they cannot be observed in practice. On the other hand, it requires the need for correction of the system (3) in order to obtain its temporarily stable solutions, either in the form of limit cycles or strange attractors. Limit cycle is a closed trajectory in 2-dimensional plane (also called a phase plane, for instance, for the velocity and coordinate, or (Q_1 : Q_2), to which any trajectory inside or outside the cycle is striving. Thus, limit cycle is a temporarily stable and attracting creature.In this study analysis is focused on the limit cycles.

Model: Attempting to find the limit cycle for the system (3) we compute second derivative of Q_1 over time t using (3) and transform it by separating similar multiplying terms:

$$\frac{d^2Q_1}{dt^2} = \alpha\dot{Q}_1 - \beta\left(\dot{Q}_1Q_2 + Q_1\dot{Q}_2\right)$$

$$= \alpha\dot{Q}_1 - \beta\dot{Q}_1Q_2 - \beta Q_1\left(k\beta Q_1Q_2 - mQ_2\right)$$

$$= \left(\alpha - \beta Q_2\right)\dot{Q}_1 - k\beta^2 Q_1^2 Q_2 + \beta m Q_1 Q_2$$

$$= \left[\alpha - \beta\frac{\alpha Q_1 - \dot{Q}_1}{\beta Q_1}\right]\dot{Q}_1 - k\beta^2 Q_1^2 \frac{\alpha Q_1 - \dot{Q}_1}{\beta Q_1}$$

$$+ \beta m Q \frac{\alpha Q - \dot{Q}_1}{\beta Q}$$

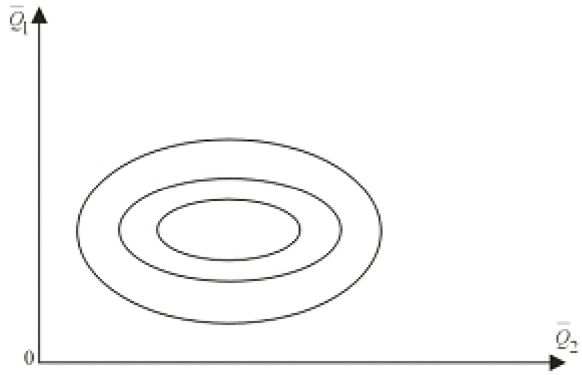

Fig. 2: Unstable cycles in Volterra's predator-prey model

$$= \frac{\dot{Q}_2}{Q_1} + k\beta Q\left(\alpha Q - Q\right) + m\left(\alpha Q - Q\right)$$

$$= \left[\frac{\dot{Q}_1}{Q_1} + k\beta Q_1 - m\right]\dot{Q}_1 - k\beta\alpha Q_1^2 + m\alpha Q_1$$

Hence, we obtained differential equation of a second order:

$$\frac{d^2Q_1}{dt^2} - \left[\frac{\dot{Q}_1}{Q_1} + k\beta Q_1 - m\right]\dot{Q}_1 + \alpha\left[k\beta Q_1^2 - mQ\right] = 0 \tag{6}$$

Equation (6) is non-linear one. Consider a case when:

$$\frac{\dot{Q}_1}{Q_1} + k\beta Q_1 - m = G; \tag{7}$$

where G = const. Since $Q_1 \neq 0$ then Eq. (7) may be transformed as follows:

$$\dot{Q}_1 + Q_1\left(k\beta Q_1 - m - G\right) = 0 \tag{8}$$

It leads to conclusion that $\overline{Q} = \dfrac{m + G}{k\beta}$ is partial solution of differential Eq. (7). Transform (8) in the following way:

$$GQ_1 - \dot{Q}_1 = \left(k\beta Q_1 - m\right)Q_1 \tag{9}$$

By multiplying both sides of Eq. (9) to and substituting it into (6) we obtain:

$$\frac{d^2Q_1}{dt^2} - \left(G + \alpha\right)\dot{Q}_1 + \alpha G Q_1 = 0 \tag{10}$$

Rewrite (10) as follows by denoting its right side as *V*:

$$\frac{d^2Q_1}{dt^2}+\alpha GQ_1=(G+\alpha)\dot{Q}_1\equiv V \tag{11}$$

For analysis of the last equation one is to use mathematic trick by introducing additional term to its right side in form of the function $V_1=\vartheta\left(\dot{Q}_1-\frac{1}{3}\dot{Q}_1^3\right)$.

Eq. (11) in this case becomes the following:

$$\frac{d^2Q_1}{dt^2}+\alpha GQ_1=(\vartheta+G+\alpha)\dot{Q}_1-\frac{\vartheta}{3}\dot{Q}_1^3 \tag{12}$$

Eq. (12) is transformed to Eq. (11) under $V_1=\vartheta\left(\dot{Q}_1-\frac{1}{3}\dot{Q}_1^3\right)=0$ i.e. under $\vartheta=0$ or Q_1 = 0; ±1.

Hence, after solving Eq. (12) we have to check if this condition is valid, and then conclusions obtained for Eq. (12) will be equally valid for Eq. (11).

One may implement the quadrature method by multiplying both sides of (12) for $\dot{Q}_1$. In this case it may be rewritten as follows:

$$\frac{1}{2}\frac{d}{dt}\left[\dot{Q}_1^2+\alpha GQ_1^2\right]=(\vartheta+G+\alpha)\dot{Q}_1^2-\frac{\vartheta}{3}\dot{Q}_1^4 \tag{13}$$

Assuming that $\vartheta > 0$, we obtain

$$(\vartheta+G+\alpha)>0,\quad (\vartheta+G+\alpha)\dot{Q}_1^2>0,\ \text{and}\ \frac{\vartheta}{3}\dot{Q}_1^4>0$$

So if the magnitude $|\dot{Q}_1|$ is small, then the right side of (12) is positive. In this case negative magnitude $-\frac{\vartheta}{3}\dot{Q}_1^4$ does not influence the sign of the right side of (12). Under bigger magnitudes of $|Q_1|$ the right side of $|\dot{Q}_1|$ (12) is negative. Hence, expression

$$E=\dot{Q}_1^2+\alpha G\dot{Q}_1^4 \tag{14}$$

is increased under small rates of changes of supplying sector's output, and it is decreased under fast rates of their changes. Under E = const an Eq. (14) makes concentric ellipses on a phase plane (Q_1 ; Q_1^2) with their centre located in the upper half of the plane.

Let's find maximum of the expression $\varphi\equiv\frac{1}{2}E(\dot{Q}_1;\dot{Q}_2)\Pi o\ \dot{Q}_1$ по Q_1 by calculating its derivative and equaling it to nil:

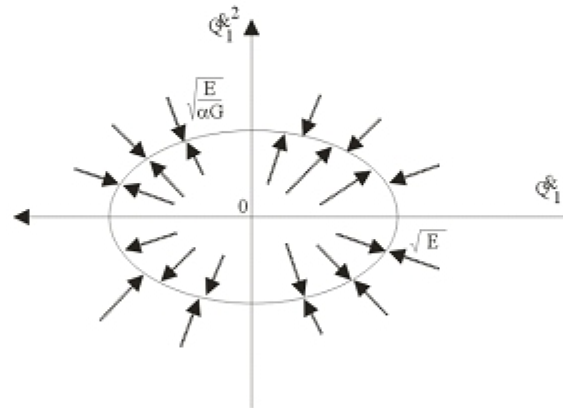

Fig. 3: Limit cycle in the "predator-prey" system

$$\frac{d\varphi}{d\dot{Q}_1}=2(\vartheta+G+\alpha)\dot{Q}_1-\frac{4}{3}\vartheta\dot{Q}_1^3=0 \tag{15}$$

It leads to the following:

$$\dot{Q}_1\left[2(\vartheta+G+\alpha)-\frac{4}{3}\vartheta\dot{Q}_1^2\right]=0 \tag{16}$$

One may see that Eq. (16) has three roots $\left[0;\pm\sqrt{\frac{3(\vartheta+G+\alpha)}{2\vartheta}}\right]$. Under $|\dot{Q}_1|<\sqrt{\frac{3(\vartheta+G+\alpha)}{2\vartheta}}$ vector field is directed to outside of the ellipse, and vice versa (Fig. 3).

Our conclusions are valid under $\vartheta\to 0$ and $\dot{Q}=0$, with the last statement meaning static production of the output. Now consider the condition $|\dot{Q}_1|<\sqrt{\frac{3(\vartheta+G+\alpha)}{2\vartheta}}$ under $\dot{Q}_1=\pm 1;|\dot{Q}|=1$, i.e. $\sqrt{\frac{3(\vartheta+G+\alpha)}{2\vartheta}}>1$ One may see that in this case $G=\frac{1}{Q}+k_i\beta Q-m\approx k_i\beta Q-m>0$. Then under $\vartheta>0$:

$$\sqrt{\frac{3(\vartheta+G+\alpha)}{2\vartheta}}=\sqrt{15+\frac{\alpha+k_i\beta Q-m}{\vartheta}}>1$$

Hence, under $\dot{Q}=\pm 1$ there comes qualitative change of the dynamics of the system, with a pair of sectors to obtain a limit cycle instead of unstable behavior of the system (3).

Construction of the trophic functions for pairs "processing-supply" sectors: Abovementioned example of the system (2) has a simplified type of non-linear interaction characterized by terms Q_1Q_2 in both equations of the (3). Closer to reality is generalized Volterra's "predator-prey" model with a trophic function $V(X)$:

$$\begin{cases} \dot{X} = \alpha X - \beta V(X)Z \\ \dot{Z} = kV(X)Z - mZ \end{cases} \tag{2}$$

It is transformed to the system (3) under $V(X) = X$. System (2) may be solved towards finding an expression for the trophic function:

$$V(X) = \frac{m[\dot{X} - \alpha X]}{\beta \dot{Z} + k[\dot{X} - \alpha X]} \tag{17}$$

Assume that a step of considered temporal interval is equal to 1 year. It allows to use time series of annual statistics for (17). Then

$$\dot{X} \equiv \lim_{\Delta t \to 0} \frac{\Delta X}{\Delta t} = \Delta X = X(t_n) - X(t_{n-1})$$

where index stands for the year of the data, i.e. $X_n \equiv X(t_n)$. In a difference form an expression (17) will be the following:

$$V(X_{n-1}) = \frac{m[(X_n - X_{n-1}) - \alpha X_{n-1}]}{\beta[Z_n - Z_{n-1}] + k[(X_n - X_{n-1}) - \alpha X_{n-1}]} \tag{18}$$

To have trophic functions constructed one has to select a pair of linked sectors, one of which shall be supplying sector for the other, and to determine coefficients α; β; k; m. Let's rewrite (2) in a difference form for the annual change of the output:

$$\begin{cases} X(t_2) - X(t_1) = \\ \qquad \alpha X(t_1) - \beta V(X(t_1))Z(t_1) \\ Z(t_2) - Z(t_1) = \\ \qquad kV(X(t_1))Z(t_1) - mZ(t_1) \end{cases} \tag{19}$$

Expression (19) shows that α- part of X is spent on reproduction in the same sector. In other words, it stands for potential capability of the assets of the sector X.

Coefficient β describes velocity of the consumption of the supply during production in the sector Z towards dynamics of the sector X. Coefficient k stands for velocity of the consumption of the supply towards dynamics of processing sector Z, which depends on delivery of supplies. Finally, coefficient m may be interpreted as amortization or wastes in sector Z, i.e. as part m from Z. Note that trophic function $V(X)$ in the system of Eq. (2) reflects capability of the sector Z to process the supply X.

To clearly understand the meaning of coefficients α; β; k one may consider the following example. Let one to have harvested 950 tons of wheat. Assume that from the total amount of the wheat 5% of seeds (coefficient $\alpha = 0.05$) is spent for next season production, if each seed of wheat produces 1 wheat ear containing 20 mature seeds in average.

Bread production sector consuming 950 tons of the wheat may be selected as the processing sector for the wheat production. In this case coefficient $\beta = 1$. If for some reason the quantity of the wheat grown was equal, for instance, to 2500 tons, while its consumption by the sector Z stays on the level of 950 tons of wheat, then coefficient $\beta = \frac{950}{2500} = 0.38$. Hence, coefficient β stands for demand of the products of sector X.

Capability of bread production may be higher than just 950 tons of wheat, and be, for instance, equal to 2000 tons. In this case processing power if the bread production sector is active for only 47,5%. Apparently, it is a value of coefficient $k = 0.47$, standing for the degree of processing industrial power of the sector Z.

Based on these conclusions, one has to obtain the following statistic and industrial information:

- output (quantity of the goods made) in sectors X and Z, i.e. temporal series X_n; Z_n
- part of the output in sector X, used in the same sector (leads to determination of α),
- part of the output in sector Z annually wasted and/or amortized (coefficient m),
- surplus of the goods produced and not consumed leading to determination of β,
- technological power of the processing sector and its load during a year or considered period (coefficient k).

We select the following pairs of sectors:

- cars production – steel production,
- cars – oil products.

Based on the logic highlighted above, time series data, rate of amortization and assumptions on the use of steel and oil in the next cycles of production (Table 1-4), one may construct two trophic functions of the sectors we have chosen (Fig. 4-5).

Analytic construction of the trophic functions: We start analytic construction of the trophic function based on Eq. (12):

Table 1. Data to compute the trophic function "cars manufacturing - steel production"

Year	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008
World cars production	53000000	56258892	58374162	56304925	58394318	60663225	64496220	66482439	69222975	73266061	70,526,531
World steel production, 1000 MT	777,328	788,969	847,670	850,345	904,053	969,992					
α	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
β	0.75	0.8	0.95	1	1	1	1	1	1	1	0.85
k	0.85	0.85	0.85	0.85	0.85	0.85	0.85	0.85	0.85	0.85	0.85
m	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
Trophic function	-0.0010792	-0.00021	0.00019	0.000378	0.000107	0.000182	0.000261	0.000241	0.00034	0.000184	0.00074172

Note:
1. Steel is needed for production of steel-rolling mills which are durable for several decades. Hence, magnitude of alfa coefficient is taken as equal to 0.05.
2. Steel consumption is changed during decades depending on the world demand. Here it is supposed to be equal to 0.95 for the growth periods; and within a range of 0.75-0.8 during recessions.
Sources: OICA; www.oica.net; World Steel Association; www.worldsteel.org.

Table 2. Data for the curve of the trophic function "cars manufacturing - steel production"

World steel production	777,328	788,969	847,670	850,345	904,053	969,992	1,069,082	1,146,686	1,251,196	1,329,719	1,351,289
Trophic function *10000	-10.792	-2.1	1.9	3.78	1.07	1.82	2.61	2.41	3.4	7.4172	1.84

Table 3. Data to compute the trophic function "cars manufacturing – oil products"

Year	1997	1998	1999	2000	2001	2002
World cars production	53117000	53000000	56258892	58374162	56304925	58394318
World production of oil products, 1000 barrel/day	62924.13	65147.42	63395.89	65856.93	65386.93	63980.75
α		0.01	0.01	0.01	0.01	0.01
β		1	1	1	1	1
k		0.85	0.85	0.85	0.85	0.85
m		0.02	0.02	0.02	0.02	0.02
Trophic function		0.259718	-0.02047	0.009961	0.007499	-0.06335
Year	2003	2004	2005	2006	2007	2008
World cars production	60663225	64496220	66482439	69222975	73266061	70526531
World production of oil products, 1000 barrel/day	67221.13	70511.73	71640.51	71715.52	71482.32	81730
α	0.01	0.01	0.01	0.01	0.01	0.01
β	1	1	1	1	1	0.95
k	0.85	0.85	0.85	0.85	0.85	0.85
m	0.02	0.02	0.02	0.02	0.02	0.02
Trophic function	0.012356	0.008576	0.003556	0.000192	0.023529	0.023529

Note:
1. Since production of the next lot of the oil products does not require presence of the previous lot of the oil products, then the magnitude of alfa coefficient is taken to be equal to 0, while here it is taken as equal to 0,01.
2. Oil products are fully consumed during 3-4 months (i.e. less than a year) after being manufactured, hence beta coefficient is taken as equal to 1.
Sources: OICA; www.oica.net; OPEC; http://www.opec.org/library/world%20oil%20outlook/WorldOilOutlook08.htm

Table 4. Data for the curve of the trophic function "cars manufacturing – oil products"

Oil products	63395.89	63980.75	65147.42	65386.93	65856.93	67221.13
Trophic function *10000	-0.02047	-0.06335	0.259718	0.007499	0.009961	0.012356
Oil products	70511.73	71482.32	71640.51	71715.52	81730	
Trophic function *10000	0.008576	0.023529	0.003556	0.000192	0.023529	

$$\frac{d^2 Q_1}{dt^2} + \alpha G Q_1 = (\vartheta + G + \alpha)\dot{Q}_1 - \frac{\vartheta}{3}\dot{Q}_1^3 \qquad (12)$$

obeying to the system of Eq. (2). Let's introduce the following denotations:

$$Q_1 = x; \dot{Q}_1 = \dot{x} = y;$$
$$\dot{y} = \ddot{x} = (\vartheta + G + \alpha)y - \frac{\vartheta}{3}y^3 - \alpha G x \qquad (20)$$

Then Eq. (12) in new variables and may be written as the following system of equations:

$$\begin{cases} \dot{x} = y; \\ \dot{y} = -\alpha G Q + (\vartheta + G + \alpha)y - \dfrac{\vartheta}{3}y^3 \end{cases} \qquad (21)$$

We have to make substitution of the variables and in such a way, that a system (21) in new variables will be identical to the system (2), i.e.:

$$\begin{cases} \dfrac{dp_1}{dt} = \alpha p_1 - \beta V(p_1)p_2 \\ \dfrac{dp_2}{dt} = kp_2 V(p_1) - mp_2 \end{cases} \qquad (22)$$

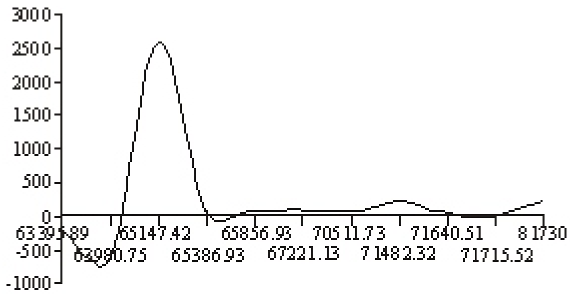

Fig. 4: Trophic function "cars – rolled steel"

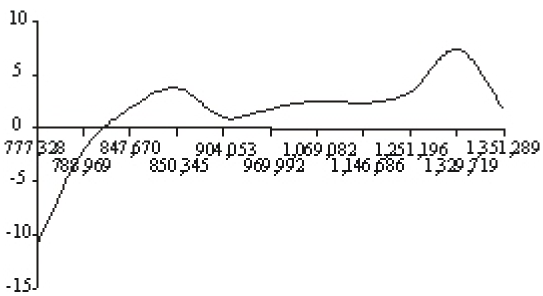

Fig. 5: Trophic function "cars – oil products"

Now we have to define new variables p_1 and p_2. Variables *x* and *y* are some functions from p_1 and p_2:

$$\begin{cases} \dot{x} = f_1(p_1;p_2) \\ \dot{y} = f_2(p_1;p_2) \end{cases} \tag{23}$$

where f_1 (p_1; p_2) and f_2 (p_1; p_2) are defined and continuously differentiable over both variables in an area $P \in R_+^2$, and variables p_1; p_2 are differentiated over *t* for $\forall t \rhd 0$, i.e. under positive time. Differentiation of the both sides in the system (23) over *t* provides the following:

$$\begin{cases} \dot{x} = \dfrac{\partial f_1}{\partial p_1} \cdot \dot{p}_1 + \dfrac{\partial f_1}{\partial p_2} \cdot \dot{p}_2 \\ \dot{y} = \dfrac{\partial f_2}{\partial p_1} \dot{p}_1 + \dfrac{\partial f_2}{\partial p_1} \dot{p}_2 \end{cases} \tag{24}$$

Assume that both functions f_1 (p_1; p_2) and f_2 (p_1; p_2) satisfy the condition:

$$\Delta = \frac{\partial f_1}{\partial p_1} \cdot \frac{\partial f_2}{\partial p_2} - \frac{\partial f}{\partial p_2} \cdot \frac{\partial f}{\partial p_1} \neq 0$$

for $\forall (p_1;p_2) \in P$ (25)

Due to the condition (25) the system (24) is uniquely solvable towards $\dot{p}_1$ and $\dot{p}_2$:

$$\begin{cases} \dot{p}_1 = \dfrac{1}{\Delta}\left[\dfrac{\partial f_2}{\partial p_2} \cdot \dot{x} - \dfrac{\partial f_1}{\partial p_2} \cdot \dot{y}\right] \\ \dot{p}_2 = \dfrac{1}{\Delta}\left[\dfrac{\partial f_1}{\partial p_1} \cdot \dot{y} - \dfrac{\partial f_2}{\partial p_1} \cdot \dot{x}\right] \end{cases} \tag{26}$$

Having in mind (23), for further discussion we denote the right part of equation (21) as a function $g(p_1; p_2)$:

$$g(p_1;p_2) = -\alpha G f_1 + (\vartheta + G + \alpha) f_2 - \frac{\vartheta}{3} f_2^3 \tag{27}$$

Using (22) and (26), we obtain the following equations:

$$\begin{cases} \dfrac{1}{\Delta}\left[\dfrac{\partial f_2}{\partial p_2} \cdot f_2 - \dfrac{\partial f_1}{\partial p_2} \cdot g\right] = \alpha p_1 - V(p_1) p_2; \\ \dfrac{1}{\Delta}\left[\dfrac{\partial f_1}{\partial p_1} \cdot g - \dfrac{\partial f_2}{\partial p_1} \cdot f_1\right] = k p_2 V(p_1) - m p_2 \end{cases} \tag{28}$$

Equations (28) provide two expressions for the trophic function $V(p_1)$:

$$\begin{cases} V(p_1) = \dfrac{1}{p_2}\left[\alpha p_1 - \dfrac{1}{\Delta}\left(\dfrac{\partial f_2}{\partial p_2} \cdot f_2 - \dfrac{\partial f_1}{\partial p_2} \cdot g\right)\right]; & (29) \\ V(p_1) = \dfrac{1}{k p_2}\left[m p_2 + \dfrac{1}{\Delta}\left(\dfrac{\partial f_1}{\partial p_1} \cdot g - \dfrac{\partial f_2}{\partial p_1} \cdot f_1\right)\right]; & (30) \end{cases}$$

Selection of the form of the functions f_1 and f_2 under the condition $\Delta \neq 0$ provides various types of the trophic function. Consider, for instance, a case when $f_1 = p_1; f_2 = \dfrac{\ln p_3}{k}$ and check the condition $\Delta \neq 0$:

$$\Delta = \frac{\partial f_1}{\partial p_1} \cdot \frac{\partial f_2}{\partial p_2} - \frac{\partial f}{\partial p_2} \cdot \frac{\partial f}{\partial p_1} = \frac{1}{k p_2} \neq 0 \tag{31}$$

Then Eq. (30) becomes the following:

$$V(p_1) = \frac{1}{kp_2}\left[mp_2 + kp_2\varepsilon\right] = \frac{m}{k} + \varepsilon$$
$$= \frac{m}{k} - \alpha G p_1 + \frac{\vartheta + G + \alpha}{k}\ln p_2 - \frac{\vartheta}{3}\left[\frac{\ln p_2}{k}\right]^3 \quad (32)$$

Under small p_2 we have $\ln p_2 \approx p_2 - 1$, which leads to the following simplification:

$$V(p_1) \approx \frac{m}{k} - \alpha G p_1 +$$
$$\frac{\vartheta + G + \alpha}{k}(p_2 - 1) - \frac{\vartheta}{3}\left[\frac{p_2 - 1}{k}\right]^3$$
$$= \left[\frac{m}{k} - \frac{\vartheta + G + \alpha}{k} + \frac{\vartheta}{3k^3}\right]$$
$$+\left[-\alpha G + \frac{\alpha}{m}(\vartheta + G + \alpha) - \frac{\alpha\vartheta}{km^2}\right]p_1 \quad (33)$$
$$+\frac{\alpha^2\vartheta}{km^2}p_1^2 - \frac{\alpha\vartheta}{3m^3}p_1^3$$

Return to initial variables taking to attention that $Q_1 = x = f_1(p_1; p_2) \equiv p_1$, and write the final form of the trophic function:

$$V(Q_1) = \left[\frac{m}{k} - \frac{\vartheta + G + \alpha}{k} + \frac{\vartheta}{3k^3}\right]$$
$$+\left[-\alpha G + \frac{\alpha}{m}(\vartheta + G + \alpha) - \frac{\alpha\vartheta}{km^3}\right]Q_1$$
$$\frac{\alpha^2\vartheta}{km^2}Q_1^2 - \frac{\alpha\vartheta}{3m^3}Q_1^3 \quad (34)$$

Example: Consider the case when
$\frac{m}{k} - \frac{\vartheta + G + \alpha}{k} + \frac{\vartheta}{3k^3} = 0$; $G=1$.

Then after regrouping the similar terms an expression (34) becomes the following:

$$V(Q_1) = \frac{\alpha\vartheta}{m}\left[\frac{1}{3k^2}Q_1 + \frac{\alpha}{km}Q_1^2 - \frac{1}{3m^2}Q_1^3\right] \quad (35)$$

Construct the curve of this function. It has a point of extremum to be obtained from the condition $\frac{dV(Q_1)}{dQ_1} = 0$.

It leads to the following equation:

$$\frac{dV(Q_1)}{dQ_1} = \frac{\alpha\vartheta}{m}\left[\frac{1}{3k^2} + \frac{2\alpha}{km}Q_1 - \frac{1}{m^2}Q_1^2\right] = 0 \quad (36)$$

or

$$Q_1^2 - \frac{2\alpha m}{k}Q_1 - \frac{m^2}{3k^2} = 0 \quad (37)$$

Solutions of this equation are the following:

$$Q_1 = \frac{m}{k}\left[\alpha \pm \sqrt{\alpha^2 + \frac{1}{3}}\right] \quad (38)$$

Since economic sense tells us that $Q_1 \triangleright 0$, then

$$Q_1^0 = \frac{m}{k}\left[\alpha + \sqrt{\alpha^2 + \frac{1}{3}}\right] \quad (39)$$

In addition, the trophic function $V(Q_1)$ has a bending point where $\frac{d^2V(Q_1)}{dQ_1^2} = 0$

$$\frac{d^2V(Q_1)}{dQ_1^2} = \frac{2}{m}\left[-\frac{1}{m}Q_1 + \frac{\alpha}{k}\right] = 0 \quad (40)$$

This leads to the conclusion that the bending point is the only one and equal to $\tilde{Q} = \frac{m\alpha}{k} \triangleright 0$ where we have conditions;

$$\frac{d^2V(\tilde{Q}_1 - \varepsilon)}{dQ_1^2} \triangleright 0; \frac{d^2V(\tilde{Q}_1 + \varepsilon)}{dQ_1^2} \triangleleft 0$$

being valid. This means that the curve of the trophic function is the following (Fig. 6):

The curve we were able to obtain shows that it can be implemented for mathematical description of certain parts of the trophic functions on Fig. 4-5, providing an option to use analytical apparatus, in particular, an expression (35).

RESULTS AND DISCUSSION

The main outcomes of the article in implementing classic "predator-prey" model to linked dynamics of the outputs are the following:

- limit cycle for Volterra's "predator-prey" model was found under quite strict constraints;
- two trophic functions "cars-rolled steel" and "cars-oil products" were constructed in the article;

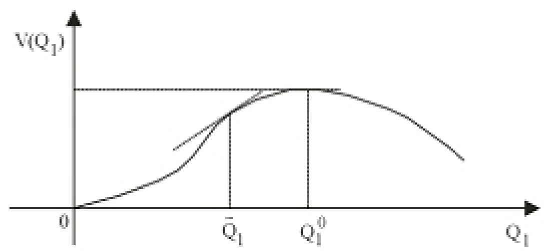

Fig. 6: Curve of analytically constructed trophic function

- we have been able to obtain expressions corresponding to the trophic functions empirically built in the article. Analytical trophic function has the limit cycle found in the article as well as a form of empirically built curves of trophic functions.

General expression of the trophic function provides a tool in estimating and using it within analysis of respective trophic links between various economic parameters.

Implementation of the technique of trophic functions in economic dynamics needs its further development to be used in middle-term economic forecasting. Based on statistic annual time-series of the output considered on a regional basis (continental; intercontinental and global scale), one may built the figures of the trophic functions of the pairs of sectors needed in economic practice. Then one may use expressions obtained in the study (e.g., (34), (35) and (40)) for the trophic functions. At their bending points the pairs of sectors will have dynamically stable relationship between the outputs of the pairs of sectors under consideration, i.e. stable demand. This seems as a reason to make a catalogue of the trophic functions, for instance, for linked outputs of sectors, either being global ones or located within some regions;

CONCLUSIONS

The tools to search and forecast the linked dynamics of economic parameters has always been of practical importance. Volterra's "predator-prey" model is one of the ways to do it, providing an insight to actual background of the linked economic dynamics using interdisciplinary non-linear approach widely accepted in physics, chemistry kinetics, biophysics etc.

Intrinsic feature of Volterra's generalized "predator-prey" model is a trophic function responsible for the type of non-linear dynamics, i.e. enabling to say how the parameters are going to behave under certain type of the trophic function. Conclusions obtained in one science are universal and successfully implemented in sciences both differing in nature of the subject and common in nature of the linked dynamics. The same is true regarding the results obtained in the article.

Although many types of the trophic dependencies have been found and analyzed up to date, one can only state that this work only starts due to huge variety of the linked dynamics observed in nature, especially in economics. As an area for future research, one may start from the point of obtained general expression for the trophic function and analytically adjust formulas to empirically observed trophic functions, as they are starting point of analysis and the way of using them in practice.

REFERENCES

Bulmer, M.G., 1976. The theory of "prey-predator" oscillations. Theor. Popul. Biol., 9(2): 137-150.

Dalimov, R.T., 2008. Modelling international economic integration and regional economic development, Trafford, pp: 234.

Freedman, H.I. and Y. Kuang, 1990. Uniqueness of limit cycles in liénard-type equations. Nonlinear Anal., 15(4): 333-338.

Gakkhar, S., B. Singh and R.K. Naji, Dynamical behavior of two "predators" competing over a single "prey". Biosystems, 90(3): 808-817.

Huang, X.C., 1990. Limit cycles in a Kolmogorov-type model and its application in immunology. Math. Comput. Model., 14: 614-617.

Leontieff, W.W., 1986. Input-Output Economics. 2nd Edn., New York: Oxford University Press, pp: 448.

Lotka, A.J., 1925. Elements of physical biology. Williams and Wilkins, Baltimore.

Milovanov, V.P., 2001. Non-Equilibrium Socio-Economic Systems: Synergetics and Self-Organization (in Russian). Moscow: Editorial URSS, pp: 264. ISBN 5-8360-0301-7.

von Neumann, J., 1945-46. A model of general economic equilibrium. Rev. Econ. Stud., 13: 1-9.

Rai, V., M. Anand and R.K. Upadhyay, 2007. Trophic structure and dynamical complexity in simple ecological models. Ecol. Complex., 4(4): 212-222.

Svirejev, Y.M. and D.O. Logofet, 1978. Stability of Biologic Communities (In Russian). Moscow: Nauka, pp: 94-112.

Volterra, V., 1931. Variations and Fluctuations of the Number of Individuals in Animal Species Living Together. In: Animal Ecology. McGraw-Hill.

Zhang, W.B., 1991. Synergetic Economics. Time and Change in Non-Linear Economics. Berlin: Springer-Verlag, pp: 261.