Research Article

A Note on Abel-Grassmann's Groupoids

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Abstract: In this study we have constructed various AG-groupoids from vector spaces over finite fields and also from finite fields by defining new operations on these structures.

Keywords: AG-groupoid, Cayley diagram, Galois field

INTRODUCTION

An Abel-Grassmann's groupoid (Protić and Stevanović, 2004), abbreviated as an AG-groupoid, is a groupoid \( S \) whose elements satisfy the invertive law:

\[(ab)c = (cb)a, \text{ for all } a, b, c \in S. \tag{1} \]

It is also called a left almost semigroup (Kazim and Naseeruddin, 1972; Mushtaq and Iqbal, 1990). In Holgate (1992) it is called a left invertive groupoid. In this study we shall call it an AG-groupoid. It has been shown in Mushtaq and Yusuf (1978) that if an AG-groupoid contains a left identity then the left identity is unique. It has been proved also that an AG-groupoid with right identity is a commutative monoid, that is, a semigroup with identity element. It is a useful non-associative algebraic structure, midway between a groupoid and a commutative semigroup.

An AG-groupoid \( S \) is medial (Kazim and Naseeruddin, 1972) that is:

\[(ab)(cd) = (ac)(bd), \text{ for all } a, b, c, d \in S. \tag{2} \]

An AG-groupoid is called an AG-band if all its elements are idempotents.

A commutative inverse semigroup \((S, \ast)\) becomes an AG-groupoid \((S, \circ)\) under the relation \(a \ast b = b \ast a^{-1}\) (Mushtaq and Yusuf, 1988).

In Stevanović and Protić (2004) a binary operation “\(\circ\)” on an AG-groupoid \( S \) has been defined as follows: for all \( x, y \in S \) there exists \( a \) such that \( x \circ y = (ax)y \). Clearly \( x \circ y = y \circ x \) for all \( x, y \in S \).

Now if an AG-groupoid \( S \) contains a left identity \( e \) then the operation \( \circ \) becomes associative, because using (1) and (2), we get:

\[(x \circ y) \circ z = (((xy)y)a)(za)((za)y) = (e(za))(za)y = (xa)((za)y) = x \circ (y \circ z). \]

Hence \((S, \circ)\) is a commutative semi group. Connection discussed above make this non-associative structure interesting and useful.

PRELIMINARIES

Here we construct AG-groupoids by defining new operations on vector spaces over finite fields. AG-groupoids constructed from finite fields are very interesting. It is well known that a multiplicative group of a finite field is a cyclic group generated by a single element. By using these generators we have drawn the Cayley diagrams for such AG-groupoids which have been constructed from finite fields. The diagrams are either bi-partite (that is, their vertices can be colored by using two minimum colors) or tri-partite (that is, they can be colored using three minimum colors).

Here we begin with the examples of AG-groupoids having no left identity.

Example 1: Let \( S = \{1, 2, 3, 4, 5, 6, 7\} \), the binary operation \( \ast \) be defined on \( S \) as follows:

<table>
<thead>
<tr>
<th>( a )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
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<th>( 6 )</th>
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<tbody>
<tr>
<td>( 1 )</td>
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<td>( 2 )</td>
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<td>7</td>
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<td>( 7 )</td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

Then \((S, \ast)\) is an AG-groupoid without left identity.
Following is an example of an AG-groupoid with the left identity.

**Example 2:** Let \( S = \{1,2,3,4,5,6,7,8\} \), the binary operation be defined on \( S \) as follows:

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 \\
3 & 4 & 5 & 6 & 7 & 8 & 1 \\
4 & 5 & 6 & 7 & 8 & 1 & 2 \\
5 & 6 & 7 & 8 & 1 & 2 & 3 \\
6 & 7 & 8 & 1 & 2 & 3 & 4 \\
7 & 8 & 1 & 2 & 3 & 4 & 5 \\
8 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

Then \((S,\cdot)\) is an AG-groupoid with left identity 7.

A graph \( G \) is a finite non-empty set of objects called vertices (the singular is vertex) together with a (possibly empty) set of unordered pairs of distinct vertices of \( G \) called edges. The vertex set is denoted by \( V(G) \), while the edge set is denoted by \( E(G) \).

A graph \( G \) is connected if every two of its vertices are connected. A graph \( G \) that is not connected is disconnected. A graph is planar if it can be embedded in the plane.

A directed graph or digraph \( D \) is a finite non-empty set of objects called vertices together with a (possibly empty) set of ordered pairs of distinct vertices of \( D \) called arcs or directed edges.

A graph \( G \) is \( n \)-partite, \( n \geq 1 \), if it is possible to partition \( V(G) \) into \( n \) subsets \( V_1, V_2, \ldots, V_n \) (called partite sets) such that every element of \( E(G) \) joins a vertex of \( V_i \) to a vertex of \( V_j \), \( i \neq j \). For \( n = 2 \), such graphs are called bi-partite graphs.

**Theorem 1:** Let \( W \) be a sub-space of a vector space \( V \) over a field \( F \) of cardinal \( 2r \) such that \( r > 1 \). Define the binary operation \( \circ \) on \( W \) as follows:

\[
u \circ v = \alpha \cdot u + \alpha v, \text{ where } \alpha, \text{ a generator of } F \setminus \{0\} \text{ and } u, v \in W.
\]

Then \((W,\circ)\) is an AG-groupoid.

**Proof:** Clearly \( W \) is closed. Next we will show that \( W \) satisfies left invertive law:

\[
(x \circ y) \circ z = \alpha^{-1}(\alpha'x + \alpha y) + \alpha z = \alpha^{-1}x + \alpha'^{-1}y + \alpha z
\]

(3)

Now:

\[
(z \circ y) \circ x = \alpha^{-1}(\alpha'z + \alpha y) + \alpha x = \alpha^{-1}z + \alpha'^{-1}y + \alpha x
\]

(4)

From (3) and (4), we get:

\[
(x \circ y) \circ z = (z \circ y) \circ x, \text{ for all } x, y, z \in W.
\]

Hence \((W,\circ)\) is an AG-groupoid.

It is not a semigroup because:

\[
x \circ (y \circ z) = \alpha'x + \alpha(\alpha' y + \alpha z) = \alpha'x + \alpha^{-1}y + \alpha^2z.
\]

(5)

(3) and (5) imply that:

\[
(x \circ y) \circ z \neq x \circ (y \circ z), \text{ for some } x, y, z \in W.
\]

Also \((W,\circ)\) is not commutative because:

\[
u \circ v = \alpha' u + \alpha v, \text{ and } v \circ u = \alpha' v + \alpha u,
\]

so

\[
u \circ v \neq v \circ u, \text{ for some } u, v \in W.
\]

Hence \((W,\circ)\) is an AG-groupoid.

**Remark 1:** An AG-groupoid \((W,\circ)\) is referred to as an AG-groupoid defined by the vector space \((V,+,\cdot)\).

**Remark 2:** If we take \( u, v \in F \), taking \( a \) as a generator of \( F \) and cardinal of \( F \) is 2, then \((F,\cdot)\) is said to be an AG-groupoid defined by Galois field.

An element \( a \) of an AG-groupoid \( S \) is called an idempotent if and only if \( a = a^2 \).

An AG-groupoid is called AG-band if all its elements are idempotents.

**CAYLEY DIAGRAMS**

A Cayley graph (also known as a Cayley colour graph and named after A. Cayley), is a graph that encodes the structure of a group.

Specifically, let \( G = \langle X \mid R \rangle \) be a presentation of the finitely generated group \( G \) with generators \( X \) and relations \( R \). We define the Cayley graph \( \Gamma = \Gamma(G,X) \) of \( G \) with generators \( X \) as:

\[
\Gamma = \langle G, E \rangle
\]

where,

\[
E = \{ u, a \cdot u \mid u \in G, a \in X \} \quad (E \text{ is the set of edges}).
\]

That is, the vertices of the Cayley graph are precisely the elements of \( G \) and two elements of \( G \) are connected by an edge if and only if some generator in \( X \) transfers the one to the other. He has proposed the use of colors to distinguish the edges associated with different generators.

**Remark 3:** If we put the value of \( r = 2 \), in remark 2, we get Galois field of order 4.

Further we need to construct a field of 4 elements, for this take an irreducible polynomial \( x^2 + x + 1 \) in \( Z_2 = \{0,1\} \). Then simple calculations yield that \( GF(2^2) = \{0,1,t, t^2\} \). The table of this field is given by:
Example 3: Using $G^2 \setminus \{0\} = F \setminus \{0\} = \{t : t^3 = 1\} = \{0, t, t^2\}$ and $u \circ v = \alpha^2 u + \alpha v$, for all $u, v \in F$ and $\alpha = t \in F$, we get the following table of an AG-groupoid:

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>0</th>
<th>1</th>
<th>$t^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$t^2$</td>
<td>1</td>
<td>0</td>
<td>$t^2$</td>
</tr>
<tr>
<td>$t^2$</td>
<td>0</td>
<td>1</td>
<td>$t^2$</td>
</tr>
</tbody>
</table>

We can draw the Cayley diagram for it as under, which is a tri-partite, planar disconnected graph (Fig. 1).

Theorem 2: Let $W$ be a sub-space of a vector space $V$ over a field $F$ of cardinal $p^s$ for some prime $p \neq 2$. Define the binary operation $\oplus$ on $W$ as follows:

$$u \oplus v = \alpha^2 u + \alpha v,$$

where $\alpha$ is a generator of $F \setminus \{0\}$ and $u, v \in W$. Then $(W, \oplus)$ is an AG-groupoid with left identity 0.

Proof: Clearly $W$ is closed. Next we will show that $W$ satisfies left invertive law:

$$(x \oplus y) \oplus z = \alpha^{-1} \left( \alpha^{s-1} x + y \right) + z = \alpha^{s-1} x + \alpha^{-1} y + z + x.$$

From (6) and (7), we get:

$$(x \oplus y) \oplus z = (z \oplus y) \oplus x, \text{ for all } x, y, z \in W.$$

Hence $(W, \oplus)$ is an AG-groupoid.

It is not a semigroup because:

$$x \oplus (y \oplus z) = \alpha^{-1} x + (\alpha^{s-1} y + z) = \alpha^{s-1} x + \alpha^{-1} y + z.$$

(6) and (8) simply that:

$$(x \oplus y) \oplus z \neq x \oplus (y \oplus z), \text{ for some } x, y, z \in W.$$

Also $(W, \oplus)$ is not commutative because:

$$u \oplus v = \alpha^2 u + \alpha v, \text{ and } v \oplus u = \alpha^{-1} v + u,$$

so $u \oplus v \neq v \oplus u$, for some $u, v \in W$.

Now:

$$0 \oplus x = \alpha^2 0 + x = x, \text{ for all } x \in W.$$

Hence $(W, \oplus)$ is an AG-groupoid with left identity 0.

Example 4: Put $p = 3$ and $n = 1$, in theorem 2, then the cardinal of $F$ is 3 and $u \circ v = \alpha u + v$, for all $u, v$ and fixed element $\alpha$ of $F$.

Obviously $F = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$ mod 3, $F \setminus \{0\} = \{1, 2\} = \{2 : 2^2 = 1\}$, here $\alpha = 2$, we get the following table of an AG-groupoid $\{0, 1, 2\}:

<table>
<thead>
<tr>
<th>$\oplus$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Now we can draw the Cayley diagram for the formed example of an AG-groupoid $(F, \circ)$, which is a bi-partite, planar disconnected graph (Fig. 2).
Example 5: Put $p = 5$ and $n = 1$, in theorem 2, then we get $|F| = 5$ and $u \otimes v = \alpha^2 u + v$.

Now clearly $GF(S) = F = Z_5 = \{0,1,2,3,4\}$ mod 5, $F \setminus \{0\} = \{2 : 2^3 = 1\}$, taking $\alpha$ as a generator which is 2, in this case, then:

$$u \otimes v = 2^2 \cdot u + v = 4 \cdot u + v.$$  

Hence we get the following AG-groupoid:

<table>
<thead>
<tr>
<th>$\otimes$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

The Cayley diagram for the above example is given by, which is a bi-partite, planar disconnected graph (Fig. 3).

Theorem 3: Let $W$ be a sub-space of a vector space $V$ over a field $F$ of cardinal $r$ such that $r > 1$. Define the binary operation $*$ on $W$ as follows:

$$u * v = au + \alpha^2 v,$$

where $\alpha$ is a generator of $F \setminus \{0\}$ and $u, v \in W$. Then $(W, *)$ is an AG-groupoid.

Proof: Clearly $W$ is closed. Next we will show that $W$ satisfies the left invertive law:

$$(x*y)z = a(ax + \alpha^2 y) + \alpha^2 z = \alpha x + \alpha^3 y + \alpha^2 z.$$  

(9)

Now:

$$(z * y)x = \alpha(ax + \alpha^2 y) + \alpha^2 x = \alpha^2 z + \alpha^3 y + \alpha^2 x = \alpha^2 x + \alpha^3 y + \alpha^2 z.$$  

(10)

From (9) and (10), we get:

$$S(x * y)z = (z * y)x, \text{ for all } x, y, z \in W.$$  

Also it is not commutative because:

$$u * v = au + \alpha^2 v, \text{ and } v * u = av + \alpha^2 u,$$

so $u * v \neq v * u$, for some $u, v \in W$.

Hence $(W, *)$ is an AG-groupoid.

Example 6: Let $|F| = 4$.

Obviously the field of order 4, is $GF(2^2) \setminus \{0\} = \{t : t^3 = 1\} = \{1, t, t^2\}$, further put $\alpha = t$ in $u * v = au + \alpha^2 v$, for all $u, v \in F$, thus obtain the following table for an AG-band $\{0,1, t, t^2\}$:

<table>
<thead>
<tr>
<th>$*$</th>
<th>0</th>
<th>1</th>
<th>$t$</th>
<th>$t^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$t^2$</td>
<td>1</td>
<td>$t$</td>
</tr>
<tr>
<td>$t$</td>
<td>1</td>
<td>$t$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$t^2$</td>
<td>1</td>
<td>$t$</td>
<td>0</td>
<td>$t^2$</td>
</tr>
</tbody>
</table>

This table now evolves the following diagram (Fig. 4).

Example 7: Let $S = \{1, 2, 3, 4\}$, the binary operation $.$ be defined on $S$ as follows:

<table>
<thead>
<tr>
<th>.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>3</td>
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<td>2</td>
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<td>1</td>
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<tr>
<td>4</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Then $(S, .)$ is an AG-band, (also given in (Protić and Stevanović, 2004)). This example is a particular form of the theorem 3.
Example 8: Let us put the value of \( r = 9 \) in theorem 3, then \( |F| = 9 \).

Now we need to construct a field of 9 elements, for this take an irreducible polynomial \( t^2 + t + 2 + 0 \) in \( \mathbb{Z}_3 = \{0, 1, 2\} \). Then simple calculations yields:

\[
GF(3^2) \setminus \{0\} = F \setminus \{0\} = \{1 + \sqrt{2}, \alpha : \alpha^8 = 1\} = \{1, 2, \sqrt{2}, 2\sqrt{2}, 2 + \sqrt{2}, 2 + 2\sqrt{2}, 1 + 2\sqrt{2}, 1 + \sqrt{2}\}.
\]

Now put the value of \( \alpha = 1 + \sqrt{2} \) in \( u * v = \alpha u + \alpha^2 v \), we get:

\[
\begin{align*}
  u * v &= (1 + \sqrt{2})u + (1 + \sqrt{2})v = (1 + \sqrt{2})u + 2\sqrt{2}v, \text{ for all } u, v \in F. \\
  v &= (1 + \sqrt{2})u + 2\sqrt{2}v.
\end{align*}
\]

Putting all the values of \( u, v \) from \( F \) in Eq. (12) we get an AG-band:

We get the following bi-partite, disconnected, planar directed graph (Fig. 5).

Remark 4: If we take finite fields instead of subspaces \( W \) of vector spaces \( V \), in theorems 1, 2 and 3, then we can make the Cayley diagrams for these AG-groupoids by using the definition of a Cayley graph.

Example 9: Let \( S = \{1, 2, 3, 4, 5, 6, 7, 8\} \), the binary operation \( \cdot \) be defined on \( S \) as follows:

\[
\begin{array}{cccccccc}
  & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 1 & 2 & 4 & 4 & 4 & 4 & 4 & 8 \\
2 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
6 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
7 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
8 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 4
\end{array}
\]

It is non-commutative and non-associative because \( 8 \neq 1 \cdot 8 = 8 \cdot 1 = 2, 2 = (2 \cdot 1) \cdot 1 \neq 2 \cdot (1 \cdot 1) = 8 \).

Example 10: Let \( S = \{1, 2, 3, 4\} \), the binary operation \( \cdot \) be defined on \( S \) as follows:

\[
\begin{array}{cccc}
  & 1 & 2 & 3 & 4 \\
1 & 1 & 2 & 3 & 4 \\
2 & 3 & 3 & 3 & 3 \\
3 & 2 & 3 & 3 & 3 \\
4 & 2 & 3 & 3 & 3
\end{array}
\]

It is non-commutative and non-associative because, \( 4 = 1 \cdot 4 \neq 4 \cdot 1 = 2 \) and \( 2 = (2 \cdot 1) \cdot 1 \neq 2 \cdot (1 \cdot 1) = 4 \). Thus \( (S, \cdot) \) is an AG-groupoid.

References


