

## Research Article

# Numerical Solutions of Mechanical Turbulent Filtration Equation Used in Mechatronics and Micro Mechanic

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**Abstract:** In this study, several novel numerical solutions are presented to solve the turbulent filtration equation and its special case called “Non-Newtonian mechanical filtration equation”. The turbulent filtration equation in porous media is a very important equation which has many applications to solve the problems appearing especially in mechatronics, micro mechanic and fluid mechanic. Many applied mechanical problems can be solved using this equation. For example, non-Newtonian mechanical filtration equation solves many filtration problems in fluid mechanic. The novel proposed discrete numerical solutions are simulated in MATLAB/simulink environment to validate the theoretically numerical solutions and proofing that the proposed numerical solutions are realizable.

**Keywords:** Fluid mechanic, mechanical filtration, porous media

## INTRODUCTION

The equation of turbulent filtration in porous media, which appears in many problems of mechanical systems, contains the p-Laplacian operator defined as the form of  $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ .

Recently some researches related to the subject have been published. A numerical method to solve the problems related to the fluid mechanics described with the equations containing the p-Laplacian operator presented in Gabriella (2008). Reaction-diffusion problems, fluid flows through certain types of porous media and Lane-Emden equations for equilibrium configurations of spherically symmetric gaseous stellar objects were described respectively in Ahmed and Sunada (1969). Also the solutions of the Emden-Fowler equation, the Einstein-Yang-Mills equations, nonlinear elasticity, glaciology and petroleum extraction appeared in Bartnik and McKinnon (1988), Otani (1984) and Schoenauer (1983). Finite difference schemes and non-standard finite difference models for partial differential equations presented in Berezin *et al.* (1998) and Damascelli *et al.* (1999). Some other researches related to the turbulent filtration equation have been published in Drabek *et al.* (1998), Drabek and Manásevich (1999), Huang (1997), Langtangen (2003), Daniel (2005), Morton and Mayers (2005), Lemos (2010) and Lemos (2012).

In this study, several novel numerical methods for solving the turbulent in porous media and non-Newtonian filtration equations are presented. Nonlinearity behavior of the turbulent in porous media and non-Newtonian filtration equations make

impossible to find explicit solutions for them. In fact, finding analytical solutions of the above two equations are impossible. So, the objective of this study is to find several novel numerical methods and schemes to find some high accurate discrete solutions for the turbulent filtration equation and its special case “Non-Newtonian mechanical filtration equation”.

## TURBULENT AND NON-NEWTONIAN FILTRATION EQUATIONS

By scaling the constants, the equation of turbulent filtration in porous media is as following (Gabriella, 2008):

$$\frac{\partial u}{\partial t} = \Delta_p (u^n) \quad (1)$$

By replacing  $n = 1$  in (1) the non-Newtonian filtration equation can be find as the following equation:

$$\frac{\partial u}{\partial t} = \Delta_p u \quad (2)$$

Consider  $u(t, x, y, z)$ , the discrete form can be described as the form of  $u_{j,k,l}^i$ , where  $i$  represents the discrete time and  $j, k, l$  the spatial coordinates. The steps of discrete values for  $t, x, y$  and  $z$  are  $h, h_1, h_2$  and  $h_3$  respectively. A grid of points in the  $R^4$  space of  $(t, x, y, z)$  is introduced, so  $\Delta u$  is a vector as following:

$$\nabla u = \left( \frac{u_{j+1,k,l}^i - u_{j,k,l}^i}{h_1}, \frac{u_{j,k,l+1}^i - u_{j,k,l}^i}{h_2}, \frac{u_{j,k,l+1}^i - u_{j,k,l}^i}{h_3} \right) \quad (3)$$

By introducing the new operator as the following equation:

$$(q_{j,k,l}^i, r_{j,k,l}^i, s_{j,k,l}^i) = |\nabla u|^{p-2} \nabla u \quad (4)$$

It can be found that:

$$(q_{j,k,l}^i, r_{j,k,l}^i, s_{j,k,l}^i) = \left( \sqrt{\left( \frac{u_{j+1,k,l}^i - u_{j,k,l}^i}{h_1} \right)^2 + \left( \frac{u_{j,k,l+1}^i - u_{j,k,l}^i}{h_2} \right)^2 + \left( \frac{u_{j,k,l+1}^i - u_{j,k,l}^i}{h_3} \right)^2} \right)^{p-2} \left( \frac{u_{j+1,k,l}^i - u_{j,k,l}^i}{h_1}, \frac{u_{j,k,l+1}^i - u_{j,k,l}^i}{h_2}, \frac{u_{j,k,l+1}^i - u_{j,k,l}^i}{h_3} \right) \quad (5)$$

It results that:

$$\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) = \nabla \cdot (q_{j,k,l}^i, r_{j,k,l}^i, s_{j,k,l}^i) \quad (6)$$

### NUMERICAL SOLUTIONS OF NON-NEWTONIAN FILTRATION EQUATION

**First numerical solution:** By considering the Non-Newtonian filtration equation described by (2) and

using (6), the first numerical solution can be earned as following:

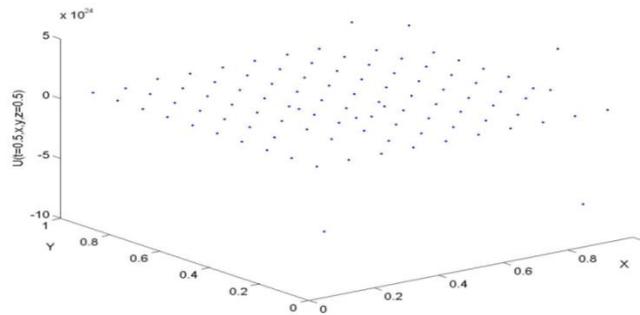
$$\frac{u_{j,k,l}^{i+1} - u_{j,k,l}^i}{h} = \frac{q_{j+1,k,l}^i - q_{j,k,l}^i}{h_1} + \frac{r_{j,k,l+1}^i - r_{j,k,l}^i}{h_2} + \frac{s_{j,k,l+1}^i - s_{j,k,l}^i}{h_3} \quad (7)$$

**Second numerical solution:** A second numerical solution can be proposed as following:

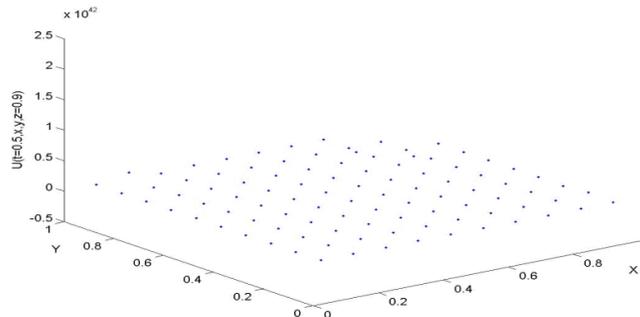
$$\frac{u_{j,k,l}^{i+1} - u_{j,k,l}^i}{h} = \frac{q_{j,k,l}^{i+1} - q_{j-1,k,l}^{i+1}}{h_1} + \frac{r_{j,k,l}^{i+1} - r_{j,k-1,l}^{i+1}}{h_2} + \frac{s_{j,k,l}^{i+1} - s_{j,k,l-1}^{i+1}}{h_3} \quad (8)$$

**Example 1:** Assume that initial condition is  $u(0, x, y, z) = x - 2y + z$  and also boundary conditions are:

$$\begin{aligned} u(t, x, y, z \leq 0) &= 2t + x - 2y \\ u(t, x, y, z \geq 1) &= 2t + x - 2y + 1 \\ u(t, x, y \leq 0, z) &= 2t + x + z \\ u(t, x, y \geq 1, z) &= 2t + x - 2 + z \end{aligned}$$



(a)



(b)

Fig. 1: (a) First numerical solution of the non-newtonian filtration equation ( $u(t = 0.5, x, y, z = 0.5)$  v.s axis  $x$  and axis  $y$ ), (b) second numerical solution of the non-newtonian filtration equation ( $u(t = 0.5, x, y, z = 0.9)$  v.s axis  $x$  and axis  $y$ )  $u(t, x \leq 0, y, z) = 2t - 2y + z$

$$u(t, x \geq 1, y, z) = 2t + 1 - 2y + z$$

Figure 1a, b shows the numerical solutions results of the Non-Newtonian filtration equation by implementing the first and second numerical solutions described by (7) and (8), respectively. The numerical solutions have been done in MATLAB using the above conditions and by considering  $p = 3$ .

**Third numerical solution:** The third numerical solution has the right-hand side as the average of the two right-hand sides of (7) and (8). It can be written as the following equation:

$$\frac{u_{j,k,l}^{i+1} - u_{j,k,l}^i}{h} = \frac{1}{2} \left( \frac{q_{j+1,k,l}^i - q_{j,k,l}^i}{h_1} + \frac{r_{j,k+1,l}^i - r_{j,k,l}^i}{h_2} + \frac{s_{j,k,l+1}^i - s_{j,k,l}^i}{h_3} \right)$$

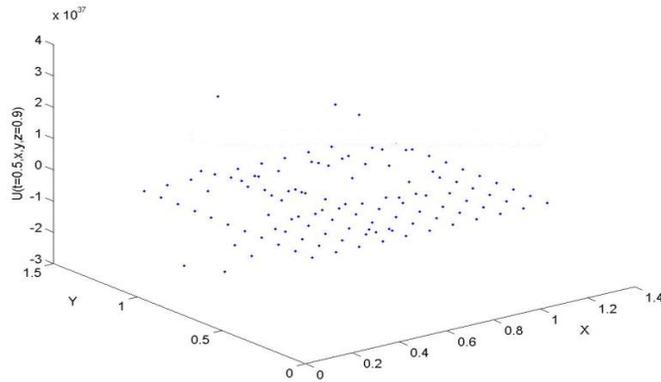
$$+ \frac{1}{2} \left( \frac{q_{j,k,l}^{i+1} - q_{j-1,k,l}^{i+1}}{h_1} + \frac{r_{j,k,l}^{i+1} - r_{j,k-1,l}^{i+1}}{h_2} + \frac{s_{j,k,l}^{i+1} - s_{j,k,l-1}^{i+1}}{h_3} \right) \tag{9}$$

**Fourth numerical solution:** The fourth numerical solution is as the following form:

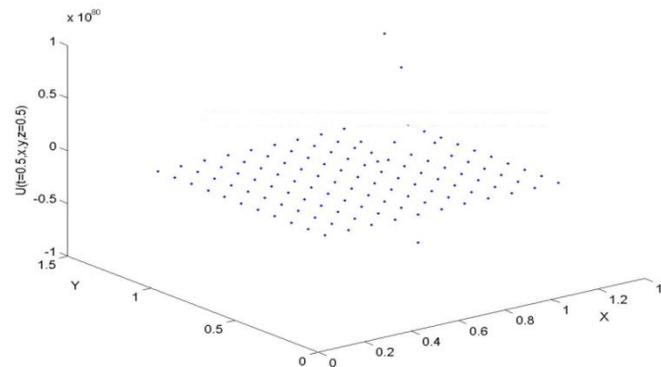
$$\frac{u_{j,k,l}^{i+1} - u_{j,k,l}^i}{h} = \frac{q_{j+1,k,l}^i - q_{j-1,k,l}^i}{2h_1} + \frac{r_{j,k+1,l}^i - r_{j,k-1,l}^i}{2h_2} + \frac{s_{j,k,l+1}^i - s_{j,k,l-1}^i}{2h_3} \tag{10}$$

**Fifth numerical solution:** The fifth numerical solution can be written as:

$$\frac{u_{j,k,l}^{i+1} - u_{j,k,l}^i}{h} = \frac{q_{j+1,k,l}^{i+1} - q_{j-1,k,l}^{i+1}}{2h_1} + \frac{r_{j,k+1,l}^{i+1} - r_{j,k-1,l}^{i+1}}{2h_2} + \frac{s_{j,k,l+1}^{i+1} - s_{j,k,l-1}^{i+1}}{2h_3} \tag{11}$$

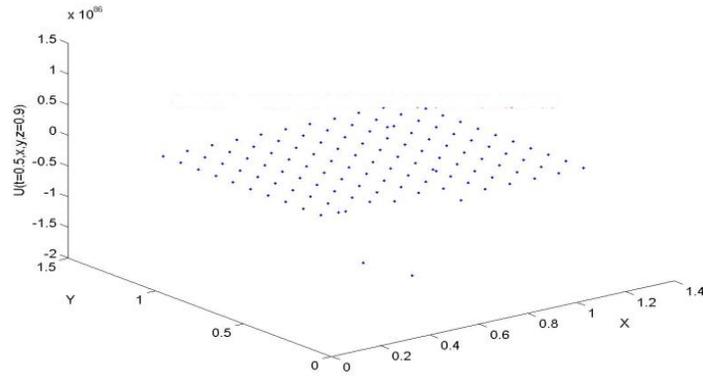


(a)

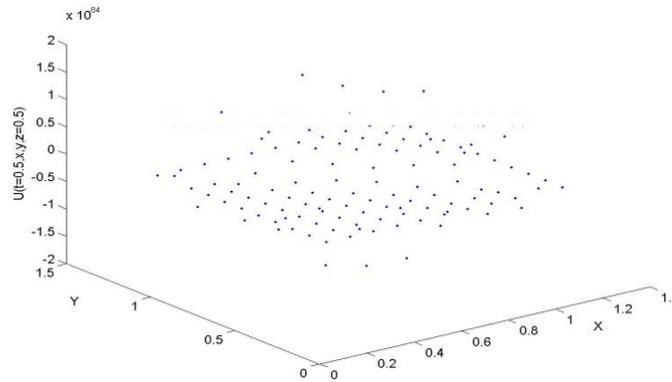


(b)

Fig. 2: (a) Fourth numerical solution of the non-newtonian filtration equation ( $u(t = 0.5, x, y, z = 0.9)$  v.s axis  $x$  and axis  $y$ ), (b) fifth numerical solution of the non-newtonian filtration equation ( $u(t = 0.5, x, y, z = 0.5)$  v.s axis  $x$  and axis  $y$ )



(a)



(b)

Fig. 3: (a) Seventh numerical solution of the non-newtonian filtration equation (u (t = 0.5, x, y, z = 0.9) v.s axis x and axis y), (b) eighth numerical solution of the non-newtonian filtration equation (u (t = 0.5, x, y, z = 0.5) v.s axis x and axis y)

**Example 2:** The results of the fourth and fifth numerical solutions of the Non-Newtonian filtration equation have been shown in Fig. 2a, b by considering  $u(1, x, y, z) = 2 + x - 2y + z$  as the limited time condition.

**Sixth numerical solution:** The sixth discrete solution is as the form of:

$$\frac{u_{j,k,l}^{i+1} - u_{j,k,l}^i}{h} = \frac{1}{2} \left( \frac{q_{j+1,k,l}^i - q_{j-1,k,l}^i + \frac{r_{j,k+1,l}^i - r_{j,k-1,l}^i}{2h_2}}{2h_1} + \frac{s_{j,k,l+1}^i - s_{j,k,l-1}^i}{2h_3} \right) + \frac{1}{2} \left( \frac{q_{j+1,k,l}^{i+1} - q_{j-1,k,l}^{i+1} + \frac{r_{j,k+1,l}^{i+1} - r_{j,k-1,l}^{i+1}}{2h_2}}{2h_1} + \frac{s_{j,k,l+1}^{i+1} - s_{j,k,l-1}^{i+1}}{2h_3} \right) \quad (12)$$

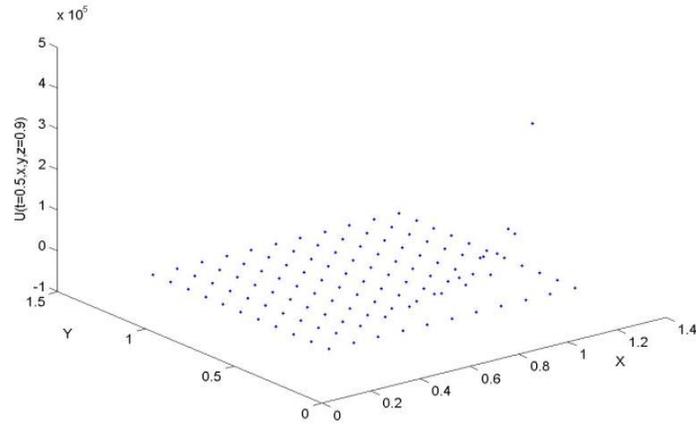
**Seventh numerical solution:** The seventh discrete solution is as the following:

$$\begin{aligned} \frac{u_{j,k,l}^{i+1} - u_{j,k,l}^i}{h} &= \frac{3q_{j+2,k,l}^i - 4q_{j+1,k,l}^i + q_{j,k,l}^i}{2h_1} \\ &+ \frac{3r_{j,k+2,l}^i - 4r_{j,k+1,l}^i + r_{j,k,l}^i}{2h_2} \\ &+ \frac{3s_{j,k,l+2}^i - 4s_{j,k,l+1}^i + s_{j,k,l}^i}{2h_3} \end{aligned} \quad (13)$$

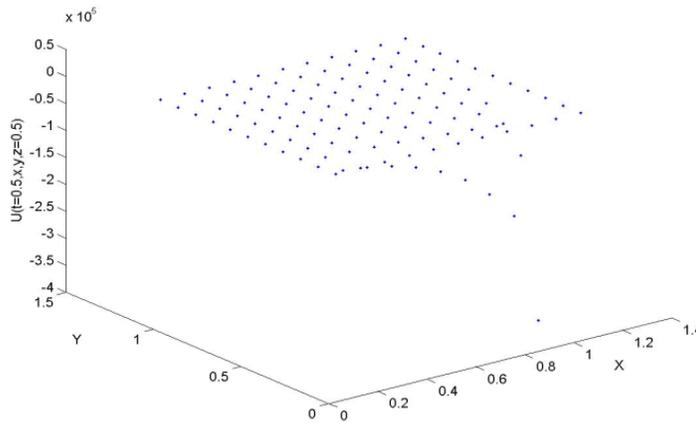
**Eighth numerical solution:** The eighth discrete solution is as the following form:

$$\begin{aligned} \frac{u_{j,k,l}^{i+1} - u_{j,k,l}^i}{h} &= \frac{3q_{j+2,k,l}^{i+1} - 4q_{j+1,k,l}^{i+1} + q_{j,k,l}^{i+1}}{2h_1} \\ &+ \frac{3r_{j,k+2,l}^{i+1} - 4r_{j,k+1,l}^{i+1} + r_{j,k,l}^{i+1}}{2h_2} \\ &+ \frac{3s_{j,k,l+2}^{i+1} - 4s_{j,k,l+1}^{i+1} + s_{j,k,l}^{i+1}}{2h_3} \end{aligned} \quad (14)$$

The results of the seventh and eighth discrete numerical solutions have been shown in Fig. 3a, b using the same conditions.



(a)



(b)

Fig. 4: (a) First numerical solution for Eq. (1) with  $n = 4$  ( $u(t = 0.5, x, y, z = 0.9)$  v.s axis  $x$  and axis  $y$ ), (b) second numerical solution for Eq. (1) with  $n = 4$  ( $u(t = 0.5, x, y, z = 0.5)$  v.s axis  $x$  and axis  $y$ )

### NUMERICAL SOLUTION OF TURBULENT FILTRATION EQUATION IN POROUS MEDIA

Consider the turbulent filtration Eq. (1), by defining  $w = u^n$ , it can be changed to the following equation:

$$(q_{j,k,l}^i, r_{j,k,l}^i, s_{j,k,l}^i) = |\nabla w|^{p-2} \nabla w = |\nabla u^n|^{p-2} \nabla u^n \quad (15)$$

so that:

$$(q_{j,k,l}^i, r_{j,k,l}^i, s_{j,k,l}^i) = \left( \sqrt{\left( \frac{w_{j+1,k,l}^i - w_{j,k,l}^i}{h_1} \right)^2 + \left( \frac{w_{j,k+1,l}^i - w_{j,k,l}^i}{h_2} \right)^2 + \left( \frac{w_{j,k,l+1}^i - w_{j,k,l}^i}{h_3} \right)^2} \right)^{p-2} \left( \frac{w_{j+1,k,l}^i - w_{j,k,l}^i}{h_1}, \frac{w_{j,k+1,l}^i - w_{j,k,l}^i}{h_2}, \frac{w_{j,k,l+1}^i - w_{j,k,l}^i}{h_3} \right) \quad (16)$$

So, similarly with the previous section, the following discrete numerical solutions can be derived.

**First numerical solution:** The first numerical solution can be earned as the following:

$$\frac{u_{j,k,l}^{i+1} - u_{j,k,l}^i}{h} = \frac{q_{j+1,k,l}^i - q_{j,k,l}^i}{h_1} + \frac{r_{j,k+1,l}^i - r_{j,k,l}^i}{h_2} + \frac{s_{j,k,l+1}^i - s_{j,k,l}^i}{h_3} \quad (17)$$

**Second numerical solution:** The second numerical solution is as the following:

$$\frac{u_{j,k,l}^{i+1} - u_{j,k,l}^i}{h} = \frac{q_{j,k,l}^{i+1} - q_{j-1,k,l}^{i+1}}{h_1} + \frac{r_{j,k,l}^{i+1} - r_{j,k-1,l}^{i+1}}{h_2} + \frac{s_{j,k,l}^{i+1} - s_{j,k,l-1}^{i+1}}{h_3} \quad (18)$$

**Third numerical solution:** The third numerical solution is as the following:

$$\frac{u_{j,k,l}^{i+1} - u_{j,k,l}^i}{h} = \frac{1}{2} \left( \frac{q_{j+1,k,l}^i - q_{j,k,l}^i}{h_1} + \frac{r_{j,k+1,l}^i - r_{j,k,l}^i}{h_2} + \frac{s_{j,k,l+1}^i - s_{j,k,l}^i}{h_3} \right) + \frac{1}{2} \left( \frac{q_{j,k,l}^{i+1} - q_{j-1,k,l}^{i+1}}{h_1} + \frac{r_{j,k,l}^{i+1} - r_{j,k-1,l}^{i+1}}{h_2} + \frac{s_{j,k,l}^{i+1} - s_{j,k,l-1}^{i+1}}{h_3} \right) \quad (19)$$

**Example 3:** The results of the first and second discrete numerical solutions of the turbulent filtration equation (Eq. (17) and (18)) have been shown in Fig. 4a, b using the same conditions and for  $n = 4$ .

### CONCLUSION

In this study, several numerical solutions were presented to solve the turbulent filtration equation and its special case which is the non-Newtonian mechanical filtration equation. The results of the proposed discrete numerical solutions were simulated using the MATLAB software to proof that the proposed numerical solutions are realizable.

### REFERENCES

Ahmed, N. and D.K. Sunada, 1969. Nonlinear flow in porous media. *J. Hydr. Div. Proc. Am. Soc. Civil Eng.*, 95: 1847-1854.  
 Bartnik, R. and J. McKinnon, 1988. Particle-like solutions of the Einstein-yang-mills equations. *Phys. Rev. Lett.*, 61: 141-148.  
 Berezin, Y.A., K. Hutter and L.A. Spodareva, 1998. Stability properties of shallow granular flows. *Int. J. Non-Linear Mech.*, 33: 647-656.  
 Damascelli, L., F. Pacella and M. Ramaswamy, 1999. Symmetry of ground states of p-Laplace equations via the moving plane method. *Arch. Ration. Mech. Anal.*, 148: 291-298.

Daniel, R.L., 2005. *Numerical Partial Differential Equations for Environmental Scientists and Engineers, a First Practical Course*. Springer, New York.  
 Drabek, P. and R. Manásevich, 1999. On the closed solution to some nonhomogeneous eigenvalue problem with p-Laplacian. *Differ. Integral Equ.*, 12: 773-782.  
 Drabek, P., P. Krejci and P. Takac, 1998. Nonlinear differential equations. *Proceeding of Seminar in Differential Equations*, Chvalatice, Czech Republic, pp: 134-141.  
 Gabriella, B., 2008. Numerical and analytic investigation of some nonlinear problems in fluid mechanics. *Comput. Simul. Modern Sci.*, 1(2): 172-178.  
 Huang, Y.X., 1997. A note on the asymptotic behavior of positive solutions for some elliptic equation. *Nonlinear Anal.*, 29: 533-541.  
 Langtangen, H.P., 2003. *Computational Partial Differential Equations, Numerical Methods and Differential Pack Programming*. Springer, New York.  
 Lemos, M.J.S.D., 2010. Simulation of turbulent combustion in porous materials with One- and two-energy equation models. *Heat Transfer Multi-Phase Mater.*, 2: 443-460.  
 Lemos, M.J.S.D., 2012. *Turbulence in Porous Media: Modeling and Applications*. Elsevier, New York.  
 Morton, K.W. and D.F. Mayers, 2005. *Numerical Solution of Partial Differential Equations: An Introduction*. Cambridge University Press, England.  
 Otani, M., 1984. A remark on certain nonlinear elliptic equations. *Proc. Fac. Sci. Tokai Univ.*, 19: 23-31.  
 Schoenauer, M., 1983. A Monodimensional Model for Fracturing. In: Fasano, A. and M. Primicerio (Eds.), *Free Boundary Problems. Theory Applications*, Pitman Research Notes in Mathematics, London, 79: 701-711.