

## Research Article

# Output-synchronization for the Different-order Uncertain Chaotic Systems via Fuzzy Sliding Control

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**Abstract:** In this study, the synchronization of the outputs for the uncertain chaotic systems with different-orders is presented. An adaptive fuzzy system has been used to approximate the uncertain nonlinear terms. The adaptive fuzzy control strategy is estimated to guarantee output-synchronization of the master-slave chaotic systems based on sliding control theorem. By using a dynamic compensator, the performance of the closed-loop systems in sliding mode is improved. The proposed controller can ensure synchronous error converges to zero. Simulation results are provided to illustrate the effectiveness of the proposed method.

**Keywords:** Chaotic system, fuzzy sliding control, output-synchronization, synchronization

## INTRODUCTION

Since the possibility of controlling chaos was proved by Pecora and Carroll (1990) in early 90', the chaos synchronization has received noticeably attention. Its many potential applications such as secure communication, biological systems, digital communication, chemical reaction and design and soon have been investigated. For chaotic synchronization, there are several methods proposed: complete synchronization (Pecora and Carroll, 1990; Agiza, 2004), phase synchronization (Rosenblum *et al.*, 1996), lag synchronization (Boccaletti and Valladares, 2000; Chen *et al.*, 2007; Miao *et al.*, 2009), generalized synchronization (Wang and Guan, 2006), modified function projective synchronization (Li and Li, 2011), state function synchronization (Li *et al.*, 2012) etc.

Synchronization is not always done between systems with the same order in natural systems (Femat and Solís-Perales, 2002). For example, the synchronization between human heart and lungs, the synchronization of human neurons in the brain (Terman *et al.*, 1998; Schafer *et al.*, 1999). Laoye *et al.* (2008) investigated reduced-order synchronization of the rigid body based on back-stepping control. Active control for the Chaos synchronization of different order system was presented (Ge *et al.*, 2006). Rodriguez *et al.* (2008) investigated Quasi-continuous high-order sliding-mode controllers for reduced-order chaos synchronization. Vincent and Guo (2009) presented a simple adaptive control for full and reduced-order synchronization of time-varying uncertain chaotic systems. Xu *et al.* (2008) investigated the full- and reduced-order synchronization of a class of time-varying uncertain systems.

Motivated with the available results, one aims at proposed a new synchronization strategy: synchronization of the outputs of the master-slave chaotic systems with different-order and uncertainties.

Compared with available results, the main contributions of our work include: firstly, presenting the output-synchronization strategy for the different-order chaotic systems; secondly, considering the parameters of the systems variable; thirdly, designing a dynamic compensator, which improve the performance of the closed-loop systems in sliding mode. Finally, an example is demonstrated to verify the effectiveness of the criterion proposed.

## METHODOLOGY

**Problem description:** Consider the master-slave system as below:

$$\dot{x} = f(x) + \Delta f(x, \sigma, t), \quad (1)$$

$$z_1 = C_1 x, \quad (2)$$

$$\dot{y} = g(y) + \Delta g(y, v, t) + D u, \quad (3)$$

$$z_2 = C_2 y. \quad (4)$$

where,  $x \in R^n$ ,  $y \in R^m$ , are states,  $u \in R^p$  is input and  $z_1, z_2 \in R^p$  are outputs ( $1 \leq p \leq \min \{n, m\}$ ). Vector functions  $f(x) \in R^n$ ,  $g(y) \in R^m$  and  $\Delta f(x, \sigma, t) \in R^n$ ,  $\Delta g(y, v, t) \in R^m$  are known nonlinear terms and the nonlinear uncertainty experienced by the systems. It is assumed that all the nonlinear functions are smooth enough.  $\sigma$  and  $v$  are parameter disturbances and satisfying

$\|\sigma\| \leq \delta_1, \|v\| \leq \delta_2$ ,  $\delta_1$  and  $\delta_2$  are positive constants.  $D \in R^{n \times p}$  is full column rank constant matrix,  $C_1 \in R^{n \times p}$  and  $C_2 \in R^{p \times m}$  are full row rank ones.

**Design of the Fuzzy Logic System (FLS):** The  $i^{\text{th}}$  fuzzy rule is written as:

$R^l$ : if  $x_1$  is  $F_1^l$ ,  $x_2$  is  $F_2^l, \dots, x_n$  is  $F_n^l$ , Then  $f$  is  $G^l$ ,  $l = 1, \dots, N$

where,  $x = [x_1, \dots, x_n]^T$  and  $f$  are the input and output of the fuzzy logic system,  $F_i^l, G^l$  are the fuzzy sets. Fuzzy logic system could be expressed as follows:

$$f(x) = \frac{\sum_{l=1}^N w_l \prod_{i=1}^n \mu_{F_i^l}(x_i)}{\sum_{l=1}^N [\prod_{i=1}^n \mu_{F_i^l}(x_i)]} \quad (5)$$

where,  $w_j$  is the adaptive parameter, the fuzzy basis function is defined as:

$$\xi_j = \frac{\prod_{i=1}^n \mu_{F_i^j}(x_i)}{\sum_{l=1}^N [\prod_{i=1}^n \mu_{F_i^l}(x_i)]} \quad (6)$$

Suppose  $w = [w_1, \dots, w_N]^T$ ,  $\xi = [\xi_1, \dots, \xi_N]^T$ , then fuzzy system could be written as:

$$f(x) = w^T \xi \quad (7)$$

**Lemma 1:**  $g(x)$  is the continuous function defined at the tight set  $\Omega$ , then  $\forall \varepsilon > 0$ , there exists the fuzzy system (7), such that:

$$\sup_{x \in \Omega} |g(x) - f(x)| \leq \varepsilon \quad (8)$$

The objective of this study is to synchronize the outputs of the different-order master-slave chaotic systems.

### MAIN RESULTS

The synchronization error of the outputs for the systems (1) and (2) is defined as  $e = z_1 - z_2$ , i.e.:

$$e = C_2 y - C_1 x \quad (9)$$

By derivative of both sides of (9), we obtain:

$$\begin{aligned} \dot{e} &= C_2 \dot{y} - C_1 \dot{x} = C_2 (g(y) + \Delta g(y, v, t) + Du) \\ &\quad - C_1 (f(x) + \Delta f(x, \sigma, t)) \end{aligned}$$

Then the error dynamics is given by:

$$\dot{e} = F + \Delta F + \bar{u} \quad (10)$$

where,

$$\begin{aligned} F &= C_2 g(y) - C_1 f(x) \\ \Delta F &= C_2 \Delta g(y, v, t) - C_1 \Delta f(x, \sigma, t) \end{aligned}$$

$$\bar{u} = C_2 Du$$

In order to design an appropriate sliding surface for the error system (10), we introduce a compensator:

$$\dot{w} = ke - w \quad (11)$$

where,  $w \in R^p$  is the state of the compensator.  $k \in R^{p \times p}$  is the matrix to be designed later.

Consider the following sliding surface:

$$s = Ce + w \quad (12)$$

where  $C \in R^{p \times p}$  will be decided later. Differentiating the sliding surface along the solution of the error system (10) and let  $s = 0$ , then  $w = -Ce$ . so we have:

$$\begin{aligned} \dot{s} &= C\dot{e} + \dot{w} = C(F + \Delta F + \bar{u}) + (ke - w) \\ &= C(F + \Delta F + \bar{u}) + (k + C)e \end{aligned} \quad (13)$$

Let  $\dot{s} = 0$ . We get an equivalent control  $\bar{u}_e$ :

$$\bar{u}_e = -(F + \Delta F) - C^{-1}(k + C)e \quad (14)$$

By substituting (14) into (10), it yields:

$$\dot{e} = -(C^{-1}k + I)e \quad (15)$$

where,  $I \in R^{p \times p}$  is identity matrix.

From the above, we obtain the conclusion as below.

**Theorem 1:** the sliding mode dynamic Eq. (15) is asymptotically stable if there exist matrices  $C, k \in R^{p \times p}$  such that:

$$C^{-1}k + k^T C^{-T} + 2I < 0 \quad (16)$$

**Proof:** Consider the Lyapunov function:

$$V_0 = e^T e \quad (17)$$

The time derivation of the  $V_0$  is:

$$\dot{V}_0 = 2e^T \dot{e} = -e^T (C^{-1}k + k^T C^{-T} + 2I)e < 0 \quad (18)$$

So the sliding mode equation is asymptotically stable. This completes the proof.

According to lemma 1, fuzzy function  $\Psi$  is used to approximate the uncertain nonlinear function  $\Delta F$ . For convenience, we set:

$$\Delta F = \psi + \varepsilon \quad (19)$$

$$\psi = [\psi_1, \dots, \psi_p]^T, \psi_i = \omega_i^T \theta_i^*$$

$$\hat{\psi} = [\hat{\psi}_1, \dots, \hat{\psi}_p]^T, \hat{\psi}_i = \omega_i^T \hat{\theta}_i$$

$$\tilde{\psi} = [\tilde{\psi}_1, \dots, \tilde{\psi}_p]^T, \tilde{\psi}_i = \omega_i^T \tilde{\theta}_i$$

where,  $\varepsilon$  is approximating error, bounded and unknown.  $\hat{\varepsilon}$  and  $\hat{\theta}$  are the estimates of  $\theta^*$  and  $\varepsilon$ , respectively.  $\theta_i^*$  is the optimal values of  $\theta_i$ ,  $\theta_i^*$  is assumed to be constant and unknown. Note that:  $\tilde{\theta} = \theta^* - \hat{\theta}$ ,  $\tilde{\varepsilon} = \varepsilon - \hat{\varepsilon}$ .

The adaptive fuzzy control law is proposed as:

$$u = -(C_2 D)^{-1} (F + u_\Delta + u_k) \quad (20)$$

where,

$$u_\Delta = \hat{\psi} + \hat{\varepsilon} + (k + C)e \quad (21)$$

$$u_k = (\mu_0 + \mu \|s\|^{\beta-1})s \quad (22)$$

and  $\mu_0, \mu \in R^{p \times p}$  satisfying:

$$C\mu_0 + \mu_0^T C^T > 0, C\mu + \mu^T C^T > 0$$

$$\lambda_1 = \min \{ \text{eig}(C\mu_0 + \mu_0^T C^T) \}$$

$$\lambda_2 = \min \{ \text{eig}(C\mu + \mu^T C^T) \}$$

Then, it is easily obtained:

$$\dot{e} = -(C^{-1}k + I)e + \tilde{\psi} + \tilde{\varepsilon} - (\mu_0 + \mu \|s\|^{\beta-1})s \quad (23)$$

Substituting (23) into (13) yields:

$$\begin{aligned} \dot{s} &= C\dot{e} + \dot{w} \\ &= C(-(C^{-1}k + I)e + \tilde{\psi} + \tilde{\varepsilon} \\ &\quad - (\mu_0 + \mu \|s\|^{\beta-1})s) + (k + C)e \\ &= C(\tilde{\psi} + \tilde{\varepsilon} - (\mu_0 + \mu \|s\|^{\beta-1})s) \end{aligned} \quad (24)$$

Adaptive law is given by:

$$\dot{\hat{\theta}} = -2 \text{diag}(w_1, \dots, w_p) C^T s \quad (25)$$

$$\dot{\hat{\varepsilon}} = -2C^T s \quad (26)$$

**Theorem 2:** Consider the sliding surface (12). Using the adaptive fuzzy control law (20), adaptive law (25) and (26), then, the error  $e$  will move toward the switching surface and reach the sliding surface  $s = 0$ . And the error system (10) is asymptotically stable with this controller.

**Proof:** Consider the following Lyapunov function:

$$V = s^T s + \frac{1}{2} \tilde{\theta}^T \tilde{\theta} + \frac{1}{2} \tilde{\varepsilon}^T \tilde{\varepsilon} \quad (27)$$

Taking derivative of both sides of (27) with respect to time yields:

$$\begin{aligned} \dot{V} &= 2s^T \dot{s} + \tilde{\theta}^T \dot{\tilde{\theta}} + \tilde{\varepsilon}^T \dot{\tilde{\varepsilon}} \\ &= 2s^T (-C(\mu_0 + \mu \|s\|^{\beta-1})s + C\tilde{\psi} + C\tilde{\varepsilon}) \\ &\quad + \tilde{\theta}^T \dot{\tilde{\theta}} + \tilde{\varepsilon}^T \dot{\tilde{\varepsilon}} \\ &= -2s^T C(\mu_0 + \mu \|s\|^{\beta-1})s + 2s^T C\tilde{\psi} + 2s^T C\tilde{\varepsilon} \\ &\quad + \dot{\tilde{\theta}}^T \tilde{\theta} + \dot{\tilde{\varepsilon}}^T \tilde{\varepsilon} \\ &= -2s^T C(\mu_0 + \mu \|s\|^{\beta-1})s + 2s^T C \begin{bmatrix} w_1^T \tilde{\theta}_1 \\ \vdots \\ w_p^T \tilde{\theta}_p \end{bmatrix} \\ &\quad + 2s^T C\tilde{\varepsilon} + \dot{\tilde{\theta}}^T \tilde{\theta} + \dot{\tilde{\varepsilon}}^T \tilde{\varepsilon} \\ &= 2s^T C \begin{bmatrix} w_1^T & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & w_p^T \end{bmatrix} \tilde{\theta} - 2s^T C(\mu_0 + \mu \|s\|^{\beta-1})s \\ &\quad + 2s^T C\tilde{\varepsilon} + \dot{\tilde{\theta}}^T \tilde{\theta} + \dot{\tilde{\varepsilon}}^T \tilde{\varepsilon} \end{aligned} \quad (28)$$

Substituting (25) and (26) into (28), we get:

$$\begin{aligned} \dot{V} &\leq -2s^T C(\mu_0 + \mu \|s\|^{\beta-1})s \\ &\leq -s^T (C\mu_0 + \mu_0^T C^T)s \\ &\quad - \|s\|^{\beta-1} s(C\mu + \mu^T C^T)s \\ &\leq -\lambda_1 \|s\|^2 - \lambda_2 \|s\|^{\beta+1} < 0 \end{aligned} \quad (29)$$

It proves the conclusion.

**Simulation studies:** In this section, we will provide an example to show the effectiveness of the proposed method.

Consider following chen chaotic system:

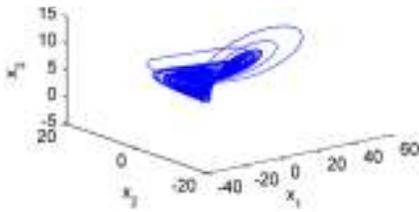


Fig. 1: The attractor of master system

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + ex_2x_3 \\ \dot{x}_2 = cx_1 - dx_1x_3 + x_2 + x_4 \\ \dot{x}_3 = -bx_3 + x_1x_2 \\ \dot{x}_4 = hx_2 \end{cases} \quad (30)$$

The system (30) exhibits chaos in Fig. 1, when the parameters  $a = 35, b = 4.8$ :

$$c = 25, d = 5, h = 12 \quad (31)$$

The chen chaotic system with uncertainty as master system is given by:

$$\dot{x} = f(x) + \Delta f(x, \sigma, t) \quad (32)$$

$$z_1 = C_1x \quad (33)$$

where,

$$f(x) = \begin{bmatrix} a(x_2 - x_1) + ex_2x_3 \\ cx_1 - dx_1x_3 + x_2 + x_4 \\ -bx_3 + x_1x_2 \\ hx_2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.4 & 0.2 & 0.1 & 0.1 \\ 0 & 0.4 & 0 & 0.2 \end{bmatrix}$$

$$\sigma = [\Delta a, \Delta b, \Delta c, \Delta r]^T, \Delta a = 0.06 \sin t,$$

$$\Delta b = 0.04 \sin t, \Delta c = 0.07 \cos t$$

$$\Delta r = 0.03 \sin t$$

$$\Delta f(x, \sigma, t) = \begin{bmatrix} \Delta a(x_2 - x_1) + \Delta ex_2x_3 \\ \Delta cx_1 + \Delta dx_1x_3 \\ \Delta bx_3 \\ \Delta hx_2 \end{bmatrix}$$

$$+ 0.1 \begin{bmatrix} 2 \sin t \\ 3 \cos 2t \\ 2 \cos 3t \\ 3 \sin 2t \end{bmatrix}$$

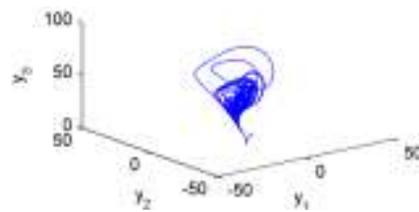


Fig. 2: The attractor of slave system

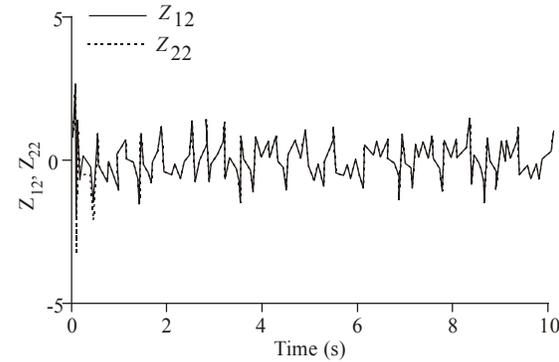
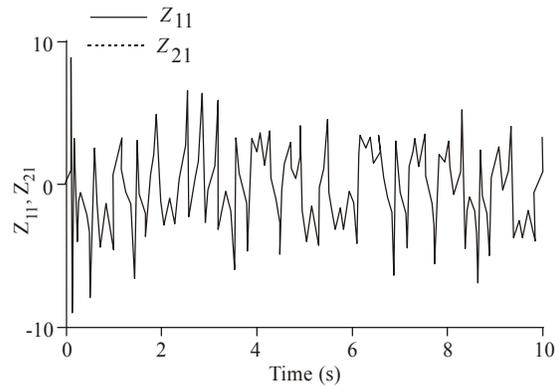


Fig. 3: The synchronization outputs

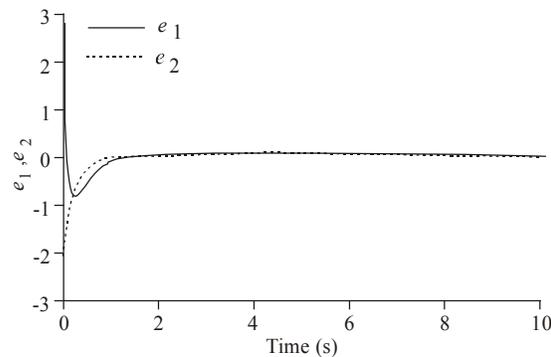


Fig. 4: The control errors

Consider following Lü chaotic system with uncertainties and input as slave one:

$$\dot{y} = g(y) + \Delta g(y, \alpha, t) + Du \quad (34)$$

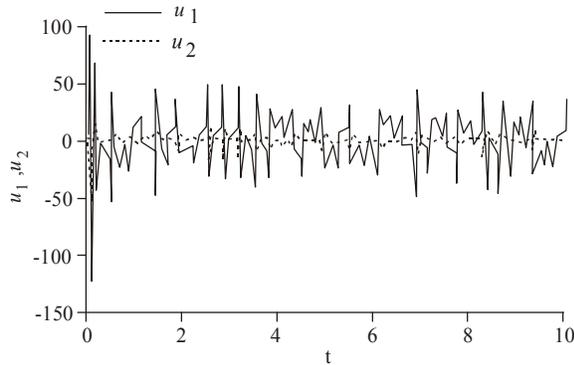


Fig. 5: The control inputs  $u_1, u_2$

$$z_2 = C_2 y \tag{35}$$

where,

$$g(y) = \begin{bmatrix} (25\alpha + 10)(y_2 - y_1) \\ (28 - 35\alpha)y_1 + (29\alpha - 1)y_2 - y_1 y_2 \\ yy - (8 + \alpha)y_3 / 3 \end{bmatrix}$$

$$\Delta g(y, v, t) = \begin{bmatrix} \Delta\alpha(y_2 - y_1) \\ \Delta\alpha(y_1 + y_2) \\ \Delta\alpha y_3 \end{bmatrix} + \begin{bmatrix} 0.01 \sin t \\ 0.02 \sin 2t \\ 0.2 \cos 4t \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0 & 0.4 & 0.2 \end{bmatrix}, D = \begin{bmatrix} 10 & 5 & 0 \\ 0 & 10 & 0 \end{bmatrix}$$

$v = \Delta\alpha = 0.01 \sin 2t$ . When parameter  $\alpha = 0.8$ ,  $u = 0$ ,  $\Delta g(y, v, t) = 0$ , the system (33) is chaos. Figure 2 shows the attractor of the Lü system (33). Now, by using control law (20), adaptive law (25), (26) and choosing  $\mu_0 = \mu = 5I$ ,  $C = I$ , The simulation results of the proposed controller are shown in Fig. 3 to 5. From Fig. 3 to 5, it is seen that synchronous error converges to zero and controller is bounded.

### CONCLUSION

In this study, a new synchronization strategy has been presented. An adaptive fuzzy controller has been constructed. The error of the output-synchronization for different-order chaotic systems with uncertainties is stable based on the designed control law and sliding mode control theorem. This synchronous method is in some sense an extension of the traditional ones. It may have some potential applications in secure communication, etc.

### ACKNOWLEDGMENT

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